To Err is Human: Implementation in Quantal Response Equilibria

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Abstract
We study the implementation problem when players are prone to make mistakes. To capture
the idea of mistakes, Logit Quantal Response Equilibrium (LQRE) is used, and we consider
a case in which players are almost rational, i.e., the sophistication level of players approaches
infinity. We show that quasimonotonicity, a small variation of Maskin Monotonicity, and
no worst alternative are the necessary conditions for restricted Limit LQRE (LLQRE) im-
plication. Moreover, these conditions are sufficient for both restricted and unrestricted
LLQRE implementations if there are at least three players and if at least one player has a
state-independent worst alternative.

Keywords: implementation, mechanisms, bounded rationality, quantal response equilibria.
JEL: C72, D70, D78.

1. Introduction

The fields of bounded rationality and mechanism design are increasingly attracting the
attention of economists. This paper contributes to solving the classical implementation
problem when players are boundedly rational. Implementation theory considers the cases in
which there are many different states of the world but in each state what is optimal for a
society (summarized by social choice rule (SCR)) is known. Even though the SCR is fixed,
there are usually states in which the social optima contradict the individual optima for some
agents. Therefore, it is unreasonable to expect society to make a choice consistent with the
SCR after a state is realized. However, a benevolent third party who is not aware of the
realized state may be able to guarantee socially optimal outcomes in each state by designing
a mechanism (a set of rules that result in an outcome based on the information sent by
this society’s members) that is played by the members once the uncertainty is resolved.
Implementation theory investigates whether any mechanism can deliver the socially optimal
outcomes in each state.

Since Hurwicz’s seminal works in 1960 and 1972, the implementation problem has been
studied from many different perspectives. However, a majority of papers assumes that
players are fully rational. But what if the players are not fully rational? This is the main
concern of this paper.

1 For more information see Jackson (2001), Maskin and Sjostrom (2002) and Serrano (2004).
In this paper, we model irrationality as simple mistakes that occur when the players evaluate their best responses: the players try to be rational, but because of their imperfect calculating ability, they may play non-optimal strategies. If some probabilistic structure is imposed on the mistakes, then the players have probabilistic responses. Assuming that the players are aware that the others are mistake-prone, one can define equilibrium as a fixed point of the players’ responses. This equilibrium is the well known Quantal Response Equilibrium (QRE) from McKelvey and Palfrey (1995).

Logit QRE (LQRE) is an QRE if mistakes are distributed iid with an extreme value distribution parameterized by $\lambda \in \mathbb{R}_+$ which we interpret here as the sophistication level. Thanks to this specification of the mistakes, the logit quantal response function has two desirable features. First, the players are more likely to make a smaller mistake than a bigger one. Second, as the sophistication level approaches infinity, the probability of a player playing a strategy not in the true best response monotonically decreases to 0. Therefore, the higher the $\lambda$, the more rational players are. In addition, if $\lambda = \infty$, then the players are fully rational, hence, any $\text{limit LQRE}$ (LLQRE) is a Nash equilibrium. Moreover, if the players are close enough to being fully rational, then any resulting LQRE is very close to one of the LLQREs.

In addition to its desirable theoretical features, LQRE seems to explain the experimental results better than the Nash equilibrium does. The original paper of McKelvey and Palfrey (1995) demonstrates the predictive ability of LQRE on several well known experiments whose results systematically deviated from the ones Nash equilibrium predicts. Since then, LQRE has been used to explain many experimental results including all-pay auctions (Anderson et al., 1998), first price auctions (Goeree et al., 2002), coordination games (Anderson et al., 2001), and information cascades (Goeree et al., 2007).

Given the theoretical and empirical plausibility of LQRE, we assume that games result in LQREs. This paper studies the implementation problem when the equilibrium concept is LLQRE (not LQRE) because the LQREs can be proxied by the corresponding LLQREs when the players are highly sophisticated. Therefore, any mechanism which implements an SCR in LLQREs will implement the SCR in LQREs with high probability.

We characterize the sufficient conditions for LLQRE implementation in environments with at least three players and in which at least one player has a state-independent worst alternative. In such environments, any SCR satisfying quasimonotonicity (QMON) –a small variation of Maskin monotonicity (Maskin, 1999)– and no worst alternative (NWA) is LLQRE implementable. Quasimonotonic SCRs satisfy the following condition: if the strict lower contour set of an SCR alternative weakly expands for every agent when going from one state to another, then the alternative is also in the SCR of the second state. On the other hand, an SCR satisfies NWA if it does not prescribe any player’s worst alternative in any state.

In the proof of the sufficiency result, we construct a mechanism that delivers each SCR alternative in each state through some strict LLQRE of the corresponding state. We should remark that this does not mean that all the LLQREs have to be strict. There can be some non-strict LLQREs in any state as long as each one of them yields an alternative prescribed by the SCR in the corresponding state. We use the above mentioned restriction because
non-strict LLQREs are sometimes not preserved\(^2\) under monotonic transformations (including affine) of the utility functions of the players. This is a highly undesirable problem for the planner who only has information about the players’ preference relations because in this case she needs to ensure that LLQRE implementation is robust to monotonic transformations of the utility functions of the players. Otherwise, some SCR alternatives might not be implemented for certain utility representations which must be avoided. In this paper, we show that strict LLQREs do not depend on the utility representations of the players’ preferences. In addition, using several examples, we illustrate the complexities of determining the conditions under which non-strict LLQREs are robust to monotonic transformations of the utility functions of the players. Consequently, for our sufficiency result, we look for a mechanism that delivers each SCR alternative in each state through some strict LLQRE of the corresponding state. If one concentrates on the LLQRE implementation in which each SCR alternative in each state is delivered by some strict LLQRE, then we also show that QMON and NWA are necessary conditions. In this sense, the current paper (almost) fully characterizes LLQRE implementation under the restriction that each SCR alternative in each state is delivered by some strict LLQRE.

The paper most closely related to ours is the one by Cabrales and Serrano (2011): They study implementation in strict Nash equilibria and find that QMON and NWA are necessary and (almost) sufficient for implementation in strict Nash equilibria. The authors furthermore conjecture that these conditions are important for implementation in environments with boundedly rational players because many equilibrium concepts in such environments are closely related to strict Nash equilibria. Indeed their conjecture holds in our setting. In this sense, this paper and Cabrales and Serrano (2011) complement each other and seem to be consistent with a “bigger picture” of implementation when the players are boundedly rational. However, the proof of our sufficiency result uses a mechanism much different than the one used in Cabrales and Serrano (2011) because not all strict Nash implementing mechanisms implement in LLQREs.\(^3\) Furthermore, our mechanism implements any given SCR in both LLQREs and strict Nash equilibria. In this sense, our mechanism is perhaps more relevant for implementation when the players are boundedly rational.

There are a handful of papers which consider the irrationality of players in implementation theory. Cabrales (1999) and Cabrales and Ponti (2000) consider implementation in existing mechanisms under learning dynamics. Cabrales and Serrano (2011) investigate the case in which the players adjust their strategies in the direction of better responses. Interestingly, QMON is again key for implementation in recurrent strategies of better response dynamics. These papers require dynamic settings, while the setting used for this paper is static. Sandholm (2005) studies simple pricing schemes used in implementing efficient SCRs in evolutionary setting. The idea that some players are completely unpredictable has been studied by Eliaz (2002). Even though in LQRE, players play in this fashion when the sophistication level approaches 0, there is a substantive difference between our paper and that of Eliaz. In his setup only some of the players make mistakes while the others are rational. In contrast, in this paper every player makes small mistakes.

\(^2\)See Example 2.6.
\(^3\)See Example 3.4.
This paper is organized as follows: Section 2 contains preliminaries. Section 3 defines LLQRE and restricted LLQRE implementations and discusses their sufficient and necessary conditions.

2. Preliminaries

The set of players is \( N = \{1, \ldots, n\} \), and the set of social alternatives is \( A \). Let \( \Theta \) be the set of states. All \( N \), \( A \) and \( \Theta \) are finite. Each player \( i \) has a utility function \( u_i : A \times \Theta \rightarrow \mathbb{R} \). The environment is \( E = (N, A, (u_i(\cdot, \theta))_{i \in N}) \) and the set of environments is \( \mathcal{E} \). The social choice rule (SCR), depending on the state, specifies the alternatives desirable to the planner. Formally, the SCR is a mapping \( F : \Theta \rightarrow 2^A \setminus \emptyset \). The players know the realized state but not the planner.

The planner designs a mechanism (game form) which is a pair \( \Gamma = ((M_i)_{i \in N}, g) \) where \( M_i \) is player \( i \)'s message (strategy) set, and \( g : \prod_{i \in N} M_i \rightarrow A \) is the outcome function mapping each message profile to an alternative. Each pair \( (E, \Gamma) \) is a game in which the set of players is \( N \), the set of strategy profiles is \( M = \prod M_i \), and the payoff function for each player \( i \) is \( u_i(g(m), \theta) \) where \( m = (m_i)_{i \in N} \) is a message profile. In the next section, we consider the notion of limit logit quantal response equilibrium — the equilibrium notion used in this paper.

2.1. Logit Quantal Response Equilibrium

Consider a game \( (E, \Gamma) \) or equivalently \( (N, (M_i)_{i \in N}, (u_i(\cdot, \theta))_{i \in N}) \). Let each \( M_i \) consist of \( K_i \) strategies, i.e., \( M_i = \{m_{i1}, \ldots, m_{iK_i}\} \). Since the state is fixed throughout this section, we exclude \( \theta \) from the notation of the utility function. Moreover, we write \( u_i(m) \) for \( u_i(g(m)) \).

Let \( \Delta_i = \{p_i \in \mathbb{R}^{K_i} : \sum_{k=1}^{K_i} p_{ik} = 1\} \) be the set of mixed strategies for player \( i \). Sometimes the notation \( p_i(m_{ik}) \) is used for \( p_{ik} \). Slightly abusing the notation, we use \( m_{ik} \) for \( p_i \) with \( p_{ik} = 1 \). We write \( \Delta = \prod \Delta_i \) and let a typical element of \( \Delta \) be \( p = (p_i)_{i \in N} \). We occasionally use the following pieces of notation: \( m_{-i} = (m_{j}^j \neq i) \), \( p_{-i} = (p_j)_{j \neq i} \), \( (m_i, m_{-i}) = m \) and \( p = (p_i, p_{-i}) \). The players’ utility functions have an expected utility form, i.e., \( u_i(p) = \sum_{m \in M} p(m) u_i(m) \) where \( p(m) = \prod_{i \in N} P_i(m_i) \).

In this model, the players try to be rational but they have an imperfect calculating ability. To model this, let us define the function \( \bar{u}_{ik} : \Delta \rightarrow \mathbb{R} \) for each \( i \) and \( m_{ik} \) as \( \bar{u}_{ik}(p) = u_i(m_{ik}, p_{-i}) \). If player \( i \) has a perfect calculating ability, then \( \bar{u}_{ik}(p) \) is her evaluation of strategy \( m_{ik} \). Let \( \bar{u}_i(p) = (\bar{u}_{ik}(p))_{k=1, \ldots, K_i} \). For each player \( i \), the mistake \( \epsilon_i \in \mathbb{R}^{K_i} \) is distributed with an CDF \( F_i \). Define player \( i \)'s evaluation of the strategies with respect to mistake structure \( \epsilon_i \) as \( \bar{u}_i(p) = \bar{u}_i(p) + \epsilon_i \). For a given realization of the mistakes, the players choose the strategy with the highest \( \bar{u}_{ik}(p) \) against \( p \in \Delta \). Because of the random structure of the mistakes, the players respond stochastically to the others’ strategies. Let us define player \( i \)'s \( ik \) response set \( R_{ik} \in \mathbb{R}^{K_i} \) for a given strategy profile \( p \) by

\[
R_{ik}(p) = \{\epsilon_i \in \mathbb{R}^{K_i} : \bar{u}_{ik}(p) + \epsilon_{ik} \geq \bar{u}_{il}(p) + \epsilon_{il} \text{ for all } l \neq j\}.
\]

\[4\]Since QRE is defined using utilities, we are using utilities instead of preferences. However, the results in this paper are robust to monotonic transformations of the utilities. Thus we can easily translate utilities into preferences.
The probability of player $i$ playing strategy $m_{ik}$ for a given mistake structure $\epsilon_i$ is

$$\sigma_{ik}(p) = \int_{R_{ik}(p)} dF_i(\epsilon_i).$$

The function $\sigma_i(p) = (\sigma_{ik}(p))_{k=1,\ldots,K_i}$ is called the quantal response function. Assuming that each player knows that the others are mistake-prone, we can define an equilibrium concept which is known as the quantal response equilibrium.

**Definition 2.1.** Let $G$ be a game and the mistake for each player $i$ be distributed with a CDF $F_i$. A quantal response equilibrium (QRE) is a mixed strategy $\pi \in \Delta$ such that

$$\pi_{ik} = \sigma_{ik}(\pi) \text{ for all } i \in N, 1 \leq k \leq K_i.$$  

As in McKelvey and Palfrey (1995), we assume that each players’ mistakes are independent of each other and follow the extreme value distribution.

**Assumption 2.2.** For each player $i$, $\epsilon_{ij}$s are independently and identically distributed with CDF $F_{ik}(\epsilon_{ik}) = \exp(-\exp(-\lambda\epsilon_{ik} - \gamma)).$

As the parameter $\lambda$ increases, the mistakes are more concentrated around 0. Therefore, we call $\lambda$ the sophistication level. With the above specificiation of the mistakes, the logit quantal response function $\sigma_i(p)$ is given by

$$\sigma_{ik}(p, \lambda) = \frac{\exp(\lambda u_{ik}(p))}{\sum_{l=1,\ldots,l_i} \exp(\lambda u_{il}(p))} \text{ for all } 1 \leq k \leq K_i.$$  

The logit quantal response function has the following desirable properties. First, the strategies with a higher expected payoff are played more frequently for any level of sophistication $\lambda \neq 0$. Secondly, as the sophistication level increases, the players play the strategies not in the best response less frequently. Therefore, when $\lambda$ converges to infinity, the logit quantal response function converges pointwise to the best response whenever it is single valued. If it is multi valued, all strategies in the best response are played with equal probabilities. Therefore, the $\lambda = \infty$ case corresponds to the case in which the players are rational.

The logit quantal response equilibrium (LQRE) is a QRE under Assumption 2.2. Let $L(\lambda, G)$ be the set of LQREs for game $G$ and sophistication level $\lambda$. Now let us consider the Limit LQRE (LLQRE) which is the solution concept for implementation in this paper.

**Definition 2.3.** Consider a finite game $G$. Then $\pi^* \in \Delta$ is a Limit LQRE if there exists $\{\pi_\tau\} \rightarrow \pi^*$ where $\pi_\tau \in L(\lambda_\tau, G)$ for some $\{\lambda_\tau\} \rightarrow \infty$. Denote $L(G)$ as the set of LLQREs.

LLQRE has the following attractive property: if the players’ sophistication level is high enough, then any resulting LQRE is very close to one of the LLQREs. Therefore, if a mechanism implements an SCR in LLQREs, then the mechanism implements the SCR in LQREs with a high probability as long as the players are sophisticated enough.

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5See McKelvey and Palfrey (1995) for the justification of this specification.
Before proceeding, we need to clarify the connection between Nash equilibria and LLQREs. It is well known that each LLQRE is a Nash equilibrium\(^6\), but it is unclear exactly which Nash equilibrium is an LLQRE. Since this information is crucial for LLQRE implementation, we study when a pure Nash equilibrium is an LLQRE.

Lemma 2.4 shows that all strict Nash equilibria are LLQREs. The key step for the proof of this is to show that any sequence of logit quantal response functions absolutely converges to a strict Nash equilibrium on a small enough of neighborhood of the equilibrium. Once this step is proved, we know that for a high enough \(\lambda\) the logit quantal response function maps a small enough neighborhood of a strict Nash equilibrium to itself. Hence, in this small neighborhood, there is a fixed point or an LQRE thanks to the Brower’s fixed point theorem.

**Lemma 2.4.** Let \(\pi^*\) be a strict Nash Equilibrium, i.e., \(u_i(\pi^*) > u_i(\pi_i, \pi_i^* - \pi_i)\) for all \(i \in N\) and \(\pi_i \neq \pi_i^*\). Then \(\pi^*\) is an LLQRE.

**Proof.** See the Appendix.

**Remark 2.5.** Strict LLQREs are invariant to (monotonic) utility transformations. This follows from Lemma 2.4 and the well known fact that pure Nash equilibria are preserved under monotonic transformations of the players’ utilities.\(^7\)

Now let us turn our attention to whether there is any relation between non strict Nash equilibria and LLQREs. In the Appendix, we identify some conditions under which non strict Nash equilibrium is not an LLQRE. However, the full characterization seems very unlikely for two reasons: (1) even weakly dominant non-strict Nash equilibrium is not necessarily an LLQRE and (2) whether a non strict Nash equilibrium is an LLQRE is highly sensitive to (affine) utility transformations. The following example demonstrates these points.

**Example 2.6.** Let \(n = 3\). The strategies and the payoffs are given in the following table.

<table>
<thead>
<tr>
<th></th>
<th>(m_1)</th>
<th>(m_2)</th>
<th>(m_1)</th>
<th>(m_2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>P3</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>P2</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>P1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>m1</td>
<td>(1,0.5,1)</td>
<td>(1,0.5,1)</td>
<td>(1,0.5,1)</td>
<td>(0,0,0)</td>
</tr>
<tr>
<td>m2</td>
<td>(1,0.5,1)</td>
<td>(0,0,0)</td>
<td>(1,0.5,1)</td>
<td>(0,0,0)</td>
</tr>
</tbody>
</table>

For this game \((m_1)_{i=1,2,3}\) is not an LLQRE, but it is if player 2’s utilities are doubled.

First consider the payoff structure shown in the table above, and let us show that \((m_1)_{i=1,2,3}\) is not an LLQRE. In contrast, suppose that \((m_1)_{i=1,2,3}\) is an LLQRE. This means that there exists a sequence \(\{p_t\} \rightarrow (m_1)_{i=1,2,3}\) such that \(p_t \in L(\lambda_t)\) for some \(\{\lambda_t\} \rightarrow \infty\). Pick some \(\lambda \in \{\lambda_t\}\) and \(p \in L(\lambda)\). Then the following equalities must be satisfied.

\[
\lambda = \frac{\ln p_{11} - \ln p_{12}}{p_{21}p_{32} + p_{22}p_{31}} = \frac{\ln p_{21} - \ln p_{22}}{0.5p_{11}p_{32} + 0.5p_{12}p_{31}} = \frac{\ln p_{31} - \ln p_{32}}{p_{11}p_{22} + p_{12}p_{21}} \quad (1)
\]

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\(^6\)For more information see McKelvey and Palfrey (1995).

\(^7\)Mixed Nash equilibria are preserved under affine transformation of the players’ utilities.
By symmetry $p_{11} = p_{31}$ at LQRE since $p_{11}$ and $p_{31}$ are around 1, so we obtain:

$$p_{11}p_{12} \ln \frac{p_{11}}{p_{12}} = p_{21}p_{12} \ln \frac{p_{21}}{p_{22}} + p_{22}p_{11} \ln \frac{p_{21}}{p_{22}}$$

When $p_{11}$ and $p_{21}$ are close to 1, for the above equality to hold it must be that (1) $p_{11}p_{12} \ln \frac{p_{11}}{p_{12}} > p_{21}p_{12} \ln \frac{p_{21}}{p_{22}}$ and (2) $p_{11}p_{12} \ln \frac{p_{11}}{p_{12}} > p_{22}p_{11} \ln \frac{p_{21}}{p_{22}}$. Then (1) yields $p_{11} > p_{21}$ while (2) gives $p_{11} < p_{21}$. They contradict each other, hence $(m_i)_{i=1,2,3}$ is not an LLQRE.

Now let us show that $(m_i)_{i=1,2,3}$ is an LLQRE when player 2’s utility or payoff is doubled. Consider a sufficiently large $\lambda$ and $p \in L(\lambda)$. Then the following equalities must be satisfied.

$$\lambda = \ln \frac{p_{11} - \ln p_{12}}{p_{21}p_{32} + p_{22}p_{31}} = \ln \frac{p_{21} - \ln p_{22}}{p_{11}p_{32} + p_{12}p_{31}} = \ln \frac{p_{31} - \ln p_{32}}{p_{11}p_{22} + p_{12}p_{21}}$$

Then by symmetry $p_{11} = p_{21} = p_{31}$. Consequently,

$$\lambda = \frac{\ln p_{11} - \ln p_{12}}{2p_{11}p_{12}}.$$ 

Observe that the right hand side (RHS) is 0 if $p_{11} = 0.5$ and approaches $\infty$ if $p_{11} \to 1$. Consequently, there must be an $p_{11}(\lambda)$ satisfying the equality above. It is easy to see that if $\lambda \to \infty$, $p_{11}(\lambda) = p_{21}(\lambda) = p_{31}(\lambda) \to 1$. This completes the proof. \(\diamondsuit\)

Example 2.6 shows that scaling of the players’ payoffs may affect whether a non-strict (pure) Nash equilibrium is an LLQRE. We believe that this is the first paper in the literature to illustrate this phenomenon.\(^8\) As a result, some non-strict (pure) LLQREs are not preserved under affine transformations of the players’ utilities. But the pure Nash equilibria are preserved under monotonic transformations of the players’ utilities. Hence, it seems very difficult (if not impossible) to determine the relation between non-strict Nash equilibria and LLQREs.

Here we remark that some non-strict LLQREs are invariant to affine utility transformations. The conditions under which this is true seem to depend on minute details of the game in question. For example, everyone playing strategy $m_{i1}$ is an LLQRE invariant to affine utility transformations if Example 2.6 is modified so that player 3 has only one strategy $m_{31}$. In the following example, we show that a slight change to the payoff structure can affect whether LLQRE is invariant to affine utility transformations.

**Example 2.7.** Let $n = 3$. The strategies and the payoffs are given in the following table.

<table>
<thead>
<tr>
<th></th>
<th>$m_1$</th>
<th>$m_2$</th>
<th>$m_1$</th>
<th>$m_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>P3</td>
<td>m1</td>
<td>m2</td>
<td>m1</td>
<td>m2</td>
</tr>
<tr>
<td>P2</td>
<td>(1, 1, 1)</td>
<td>(a, 1, 0)</td>
<td>(0, 0, 1)</td>
<td>(0, 0, 0)</td>
</tr>
<tr>
<td>P1</td>
<td>m1</td>
<td>m2</td>
<td>m1</td>
<td>m2</td>
</tr>
<tr>
<td>m1</td>
<td>(1, 0, 0)</td>
<td>(0, 0, 0)</td>
<td>(0, 0, 0)</td>
<td>(1, 1, 1)</td>
</tr>
<tr>
<td>m2</td>
<td>(0, 0)</td>
<td>(0, 0)</td>
<td>(0, 0)</td>
<td>(0, 0)</td>
</tr>
</tbody>
</table>

\(^8\)This phenomenon is present in a large class of games. The proof of this statement for 3 player games can be provided upon request.
For this game if \(a = 0\), then \((m_1)_{i=1,2,3}\) is an LLQRE invariant to affine utility transformations. However, if \(a > 0\), \((m_1)_{i=1,2,3}\) is not an LLQRE.

For each player consider an arbitrary affine transformation of the payoffs, \(\tilde{u}_i = \alpha_i u_i + \beta_i\) where \(\alpha_i > 0\). For \((m_1)_{i=1,2,3}\) to be an LLQRE, there must exist a sequence \(\{p_t\} \to (m_1)_{i=1,2,3}\) such that \(p_t \in L(\lambda_t)\) for some \(\{\lambda_t\} \to \infty\). Pick some \(\lambda \in \{\lambda_t\}\) and \(p \in L(\lambda)\). Then the following conditions must be satisfied.

\[
\lambda = \frac{\ln p_{11} - \ln p_{12}}{\alpha_1 (a p_{21} p_{32} + p_{22} p_{32})} = \frac{\ln p_{21} - \ln p_{22}}{\alpha_2 p_{12} p_{32}} = \frac{\ln p_{31} - \ln p_{32}}{\alpha_3 p_{12} p_{22}}
\]

As a result the following two conditions must be satisfied:

\[
\begin{align*}
\alpha_2 (\ln p_{11} - \ln p_{12}) p_{12} &= \alpha_1 (\ln p_{21} - \ln p_{22}) (a p_{21} + p_{22}) \\
\alpha_2 (\ln p_{31} - \ln p_{32}) p_{32} &= \alpha_3 (\ln p_{21} - \ln p_{22}) p_{22}
\end{align*}
\]

For the \(a > 0\) case, the first equality cannot be satisfied when \(p_{11} \to 1\) and \(p_{21} \to 1\) because the left hand side (LHS) converges to 0 while the RHS converges to \(\infty\).\(^9\)

If \(a = 0\), then the RHSs of the equalities above must be close to 0 when \(p_{21}\) is sufficiently close to 1. In addition, the RHSs are continuous and approach 0 when \(p_{22} \to 0\) where \(i = 1, 3\). Hence, for any \(p_{22}\) close enough to 1, there must exist \(p_{11}\) and \(p_{31}\) such that the equalities above hold. Observe that these values must converge to 1 if \(p_{21}\) converges to 1. Furthermore, observe that \(\lambda\), for which Equality 2 holds, converges to \(\infty\) when \(p_{i1} \to 1\) for each \(i\). \(\diamond\)

3. LLQRE and Restricted LLQRE Implementations

In this section, we introduce the concepts of LLQRE and restricted LLQRE implementations and identify the necessary and sufficient conditions for (restricted) LLQRE implementation. Before we start, let us remark that we focus only on implementation in pure LLQREs, which is somewhat restrictive.

For a given mechanism \(\Gamma = ((M_i)_{i \in \mathbb{N}}, g)\), let \(L(\lambda, \Gamma, \theta)\) be the set of LQREs of \(\langle E, \Gamma \rangle\) when the sophistication level of the players is \(\lambda\). In addition, let \(L(\Gamma, \theta)\) be the set of pure LLQREs of \(\langle E, \Gamma \rangle\). Now we are ready to define the LLQRE implementation.

**Definition 3.1.** An SCR \(F\) is implementable in LLQREs if there exists a mechanism\(^{10}\) \(\Gamma = ((M_i)_{i \in \mathbb{N}}, g)\) such that \(g(L(\Gamma, \theta)) = F(\theta)\) for each \(E \in \mathcal{E}\).

In the definition above, if a mechanism implements a given SCR, then it must satisfy two requirements: (1) in each state, each SCR alternative must be reached by some LLQRE and (2) in each state, each LLQRE of the mechanism must deliver some SCR alternative.

Here let us discuss the mechanisms that deliver some SCR alternatives only through non-strict LLQREs. It turns out that such mechanisms may succeed in LLQRE implementation for some utility representations of the preferences but fail for some other utility representations. We illustrate this point with the following example.

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\(^9\) Using the L’Hospital’s rule one obtains that \(\lim_{x \to 0} (-\ln x)^2 = 0\)

\(^{10}\) The message space must be finite. Otherwise the probability of playing any specific strategy is 0 at any LQRE regardless of \(\lambda\), so this probability remains 0 at any LLQRE.
Example 3.2. Suppose there are 3 players \{1, 2, 3\}, two states \{\theta, \theta'\} and four alternatives \{a, b, c, d\}. The utilities of player \(i = 1, 2, 3\) in state \(\hat{\theta} = \theta, \theta'\) is given as follows:

<table>
<thead>
<tr>
<th></th>
<th>(\theta)</th>
<th>(\theta')</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>0 0 0</td>
<td>2 2 2</td>
</tr>
<tr>
<td>b</td>
<td>2 2 2</td>
<td>1 1 1</td>
</tr>
<tr>
<td>c</td>
<td>0 0 0</td>
<td>1 1 1</td>
</tr>
<tr>
<td>d</td>
<td>0 0 0</td>
<td>0 0 0</td>
</tr>
</tbody>
</table>

Suppose \(F(\theta) = \{b\}\) and \(F(\theta') = \{a, b\}\). Consider the mechanism \(\Gamma = (M, g)\) in which \(M_i = \{m_{i1}, m_{i2}, m_{i3}\}\) for all \(i\), and \(g\) is as follows. If each player \(i\) sends message \(m_{i1}\), then \(b\) is implemented. If each player \(i \neq j\) sends \(m_{i1}\) while player \(j = 1, 2, 3\) alone sends \(m_{j2}\), then \(c\) is implemented. If each player \(i\) sends \(m_{i3}\), then \(a\) is implemented. In all other cases \(d\) is implemented.

Claim: Mechanism \(\Gamma\) implements \(F\) in LLQREs. However, mechanism \(\Gamma\) does not implement \(F\) in LLQREs if player 2’s payoffs are scaled down by two.

Proof of the Claim. Step 1. \(\Gamma\) implements \(F\) in LLQREs.

Consider state \(\theta\). Here the players play the game in which the payoffs are:

\[
\begin{array}{ccc}
\text{P3} & \\
\text{m1} & \text{m2} & \text{m3} \\
\text{P2} & \\
\hline
\text{P1} & & \\
\text{m1} & 2 & 0 & 0 \\
\text{m2} & 0 & 0 & 0 \\
\text{m3} & 0 & 0 & 0 \\
\end{array}
\begin{array}{ccc}
\text{m1} & \text{m2} & \text{m3} \\
\text{P2} & \\
\hline
\text{P1} & & \\
\text{m1} & 0 & 0 & 0 \\
\text{m2} & 0 & 0 & 0 \\
\text{m3} & 0 & 0 & 0 \\
\end{array}
\begin{array}{ccc}
\text{m1} & \text{m2} & \text{m3} \\
\text{P2} & \\
\hline
\text{P1} & & \\
\text{m1} & 0 & 0 & 0 \\
\text{m2} & 0 & 0 & 0 \\
\text{m3} & 0 & 0 & 0 \\
\end{array}
\]

It is easy to see that \((m_{i1})_{i=1,2,3}\) is the only LLQRE in state \(\theta\).

Now let us consider state \(\theta'\). Here the players play the game in which the payoffs are:

\[
\begin{array}{ccc}
\text{P3} & \\
\text{m1} & \text{m2} & \text{m3} \\
\text{P2} & \\
\hline
\text{P1} & & \\
\text{m1} & 1 & 1 & 0 \\
\text{m2} & 1 & 0 & 0 \\
\text{m3} & 0 & 0 & 0 \\
\end{array}
\begin{array}{ccc}
\text{m1} & \text{m2} & \text{m3} \\
\text{P2} & \\
\hline
\text{P1} & & \\
\text{m1} & 1 & 0 & 0 \\
\text{m2} & 0 & 0 & 0 \\
\text{m3} & 0 & 0 & 0 \\
\end{array}
\begin{array}{ccc}
\text{m1} & \text{m2} & \text{m3} \\
\text{P2} & \\
\hline
\text{P1} & & \\
\text{m1} & 0 & 0 & 0 \\
\text{m2} & 0 & 0 & 0 \\
\text{m3} & 0 & 0 & 2 \\
\end{array}
\]

Clearly, every player \(i\) sending \(m_{i3}\) is a strict Nash equilibrium, hence this is an LLQRE. Furthermore, Example 3.15 in the Appendix shows that every player \(i\) sending \(m_{i1}\) is an LLQRE in the game above. Therefore, \(\Gamma\) implements \(F\) in LLQREs.

Step 2. \(\Gamma\) does not implement \(F\) in LLQREs if player 2’s payoffs are scaled down by two.

We need to show that every player \(i\) sending \(m_{i1}\) is not an LLQRE in state \(\theta'\). On the contrary, suppose \((m_{i1})_{i=1,2,3}\) is an LLQRE; this means that there exists a sequence \(\{p_i\} \rightarrow (m_{i1})_{i=1,2,3}\)
such that \( p_t \in L(\lambda_t, \Gamma, \theta') \) for some \( \{\lambda_t\} \to \infty \). For any \( p \in \{p_t\} \), the following conditions must be satisfied.

\[
\lambda = \frac{\ln p_{11} - \ln p_{12}}{p_{21}p_{32} + p_{22}p_{31}} = \frac{\ln p_{21} - \ln p_{22}}{p_{11}p_{32}0.5 + p_{12}p_{31}0.5} = \frac{\ln p_{31} - \ln p_{32}}{p_{11}p_{22} + p_{12}p_{21}}
\]

Observe that the equality above is identical to Equality 1 in Example 2.6. Hence, the rest of the proof is the same as the one of Example 2.6.

Implementation in any equilibrium notion should be independent of the utility representations of the players’ preferences which may not be the case for LLQRE implementation as the example above demonstrates. To avoid this problem, one needs to know the conditions that guarantee the invariance of non-strict LLQREs to utility representations if there is no restriction on the implementation mechanisms. However, these conditions, as already noted in the previous section, could be dependent on the minute details of the game in question, e.g., the payoff structure, the number of players, and the number of strategies that each player has. Hence, the usefulness of such conditions in a broad implementation setting like ours seems doubtful as one has to work with very general mechanisms. Nonetheless, we acknowledge that the necessary conditions for LLQRE implementation could be weaker than the conditions we identify in this paper.

Based on the discussion above, considering the mechanisms that deliver each SCR alternative through at least one strict LLQRE is perhaps a sensible way to ensure that LLQRE implementation is robust under different utility representations of the players’ preferences. Specifically, we look for the mechanisms that satisfy (1) each SCR alternative is delivered through at least one strict LLQRE and (2) any LLQRE (strict or non-strict) yields an SCR alternative. Requirement (1) ensures that each SCR alternative in each state is delivered by some LLQRE without depending on utility representations (Lemma 2.4). For some utility representations, there might be non-strict LLQREs, but these should not cause any harm as requirement (2) ensures that they deliver SCR alternatives.

**Definition 3.3.** Social choice rule \( F \) is restricted LLQRE (RLLQRE) implementable if there is a mechanism \( \Gamma = ((M_i)_{i \in N}, g) \), such that

1. for every environment \( E \in \mathcal{E} \) and for any \( a \in F(\theta) \), there exists \( m^* \in L(\Gamma, \theta) \) such that \( g(m^*) = a \) and every player’s best response to \( m^* \) is single valued
2. if \( m^* \in L(\Gamma, \theta) \), then \( g(m^*) \in F(\theta) \)

**3.1. RLLQRE Implementation vs. Strict Nash Implementation**

The definition of RLLQRE implementation requires that each SCR alternative is delivered by a strict LLQRE. Given that each strict LLQRE is also a strict Nash equilibrium, let us pause here to highlight the differences between RLLQRE implementation and strict Nash implementation, which is studied in Cabrales and Serrano (2011).

First of all let us remark that implementation in strict LLQREs is equivalent to implementation in strict Nash equilibria because any mechanism that implements a given SCR in one of these concepts must also implement it in the other concept thanks to Lemma 2.4. However, it is important to emphasize that RLLQRE implementation has a more stringent requirement than strict Nash (or equivalently strict LLQRE) implementation: each LLQRE
must deliver an SCR alternative in RLLQRE implementation while each strict Nash equilibrium (or equivalently each strict LLQRE) must deliver an SCR alternative in strict Nash implementation. For this reason, not all strict Nash implementing mechanisms succeed in RLLQRE implementation. To clarify this, let us consider some mechanism that implements a given SCR in strict Nash equilibria. The definition of strict Nash implementation does not rule out the possibility of the mechanism having a non-strict Nash equilibrium that delivers a non-SCR alternative. If this non-strict Nash equilibrium is an LLQRE, then the mechanism fails in RLLQRE implementing. We demonstrate this point in Example 3.4. First we need 2 more pieces of notation: let $SNE(\Gamma, \theta)$ and $SL(\Gamma, \theta)$ denote the respective sets of strict Nash equilibria and strict LLQREs of game $\langle E, \Gamma \rangle$.

**Example 3.4.** Suppose there are three players $\{1, 2, 3\}$, two states $\{\theta, \theta'\}$ and four alternatives $\{a, b, c, w\}$. The utility function of player $i = 1, 2, 3$ in state $\theta = \theta, \theta'$ is as follows:

<table>
<thead>
<tr>
<th></th>
<th>$\theta$</th>
<th></th>
<th>$\theta'$</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>P1</td>
<td>P2</td>
<td>P3</td>
<td>P1</td>
<td>P2</td>
<td>P3</td>
</tr>
<tr>
<td>$a$</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>$b$</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$c$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$w$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Suppose $F(\theta) = \{a, b\}$ and $F(\theta') = \{a\}$. Consider the following mechanism $\Gamma = (M, g)$: For each player $i$, $M_i = \{m_{i1}, m_{i2}, m_{i3}\}$. The outcome function $g$ is as follows. If each player $i$ sends message $m_{i1}$, then $b$ is implemented. If each player $i \neq j$ sends $m_{i1}$ while player $j = 1, 2, 3$ alone sends $m_{j2}$, then $c$ is implemented. If each player $i$ sends $m_{i3}$, then $a$ is implemented. In all other cases $w$ is implemented.

**Claim:** Mechanism $\Gamma$ implements $F$ in strict Nash equilibria but not in RLLQREs.

**Proof of the Claim** Step 1. $\Gamma$ implements $F$ in strict Nash equilibria.

First let us consider state $\theta$. The players play the game in which each player’s payoffs are described as follows:

<table>
<thead>
<tr>
<th>P1</th>
<th>P2</th>
<th>P3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m_1$</td>
<td>$m_1$</td>
<td>$m_1$</td>
</tr>
<tr>
<td>$m_2$</td>
<td>$m_2$</td>
<td>$m_2$</td>
</tr>
<tr>
<td>$m_3$</td>
<td>$m_3$</td>
<td>$m_3$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>P1</th>
<th>P2</th>
<th>P3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m_1$</td>
<td>$m_1$</td>
<td>$m_1$</td>
</tr>
<tr>
<td>$m_2$</td>
<td>$m_2$</td>
<td>$m_2$</td>
</tr>
<tr>
<td>$m_3$</td>
<td>$m_3$</td>
<td>$m_3$</td>
</tr>
</tbody>
</table>

It is straightforward to see that $(m_{i1})_{i=1,2,3}$ and $(m_{i3})_{i=1,2,3}$ are the only strict Nash equilibria in state $\theta$. Clearly, $(m_{i1})_{i=1,2,3}$ results in $b$ and $(m_{i3})_{i=1,2,3}$ in $a$. Consequently, $F(\theta) = SNE(\Gamma, \theta)$.

Now let us consider state $\theta'$. Here the players play the game in which the payoffs are given in Table 3. It is also straightforward to see that $(m_{i3})_{i=1,2,3}$ is the only strict Nash equilibrium in state $\theta'$. Observe that $(m_{i3})_{i=1,2,3}$ results in $a$. Consequently, $F(\theta') = SNE(\Gamma, \theta')$. 
Step 2. Γ does not RLLQRE implement $F$.

Thanks to Example 3.15 in the Appendix, we know that $(m_{i1})_{i=1,2,3}$, which results in $b$, is an LLQRE in state $\theta'$. Hence, Γ does not RLLQRE implement $F$.

Here let us remark that any RLLQRE implementable SCR is implementable in strict Nash equilibria. To see this consider a mechanism that RLLQRE implements a given SCR which by definition must deliver each SCR alternative through some strict LLQRE. Then by Lemma 2.4, the mechanism also must deliver eachSCR alternative through some strict Nash equilibrium. Now all that remains to show is that the mechanism has no “bad” strict Nash equilibrium in any state, i.e, it delivers some non-SCR alternative. But if a “bad” strict Nash equilibrium existed, it would also have been a strict LLQRE thanks to Lemma 2.4. Hence, the mechanism would fail in RLLQRE implementing the SCR.

The discussion above and Example 3.4 mean that RLLQRE implementation is not less demanding than strict Nash implementation. However, as we will see in the next section, RLLQRE implementation turns out to be not more demanding than strict Nash implementation. In other words, the two implementation notions are equally demanding.

3.2. The Necessary and Sufficient Conditions

In this section we present the key conditions for RLLQRE (and LLQRE) implementation. The same conditions were shown to be relevant for strict Nash implementation independently in Cabrales and Serrano (2011).

Definition 3.5. SCR $F$ satisfies quasimonotonicity (QMON) if whenever $a \in F(\theta)$ and $a \notin F(\theta')$ for some $a$, $\theta$ and $\theta'$, there exists a player $i \in N$ and an alternative $a_i \in A$ such that

$$u_i(a_i, \theta) < u_i(a, \theta) \text{ and } u_i(a_i, \theta') \geq u_i(a, \theta')$$

(4)

For each triplet $(a, \theta, \theta')$ satisfying $a \in F(\theta)$ and $a \notin F(\theta')$, let $I(a, \theta, \theta')$ be the set players who have some alternative satisfying condition 4. In addition, for each $i \in I(a, \theta, \theta')$ let $A_i(a, \theta, \theta')$ be the set of alternatives satisfying condition 4.

QMON is closely related to Maskin monotonicity which is known to be the necessary and almost sufficient condition for Nash implementation.

Definition 3.6. SCR $F$ satisfies Maskin monotonicity (MON) if whenever $a \in F(\theta)$ and $a \notin F(\theta')$ for some $a$, $\theta$ and $\theta'$, there exists a player $i \in N$ and an alternative $a_i \in A$ such that

$$u_i(a_i, \theta) \leq u_i(a, \theta) \text{ and } u_i(a_i, \theta') > u_i(a, \theta')$$

(5)

Even though QMON is very similar to MON, there are some interesting SCRs that are quasimonotonic but not monotonic and vise versa. For example, weak pareto correspondence is well known to be monotonic, yet is not necessarily quasimonotonic. On the other hand, strong pareto correspondence is well known to not satisfy MON, however, it does satisfy QMON with a slight restriction on the preferences of the players.\(^\text{11}\)

Now we present the final condition required for RLLQRE implementation.

\(^\text{11}\)See Cabrales and Serrano (2011) or the working paper version of this paper (Tumennasan, 2011).
Definition 3.7. SCR $F$ satisfies no worst alternative (NWA) property if $a \in F(\theta)$ implies that for any $i$ there exists an alternative $b$ such that $u_i(b, \theta) < u_i(a, \theta)$. We use the notation $A_i(a, \theta)$ to denote the set $\{b : u_i(b, \theta) < u_i(a, \theta)\}$.

This property says that the SCR must not prescribe any player’s worst alternative in any state. In many situations NWA is satisfied naturally. For example, in an exchange economy setting consider the Paretian SCR which prescribes some consumption greater than the subsistence level of consumption, $\epsilon > 0$, to every player in every state. This SCR satisfies NWA as the 0 consumption is worse than any alternative in the SCR.

3.3. Main Results

Now we are ready to present the necessity and sufficiency results for RLLQRE implementation. For the necessity result, it is useful to establish that RLLQRE implementation is equivalent to double implementation in strict LLQREs and LLQREs.

Lemma 3.8. Any RLLQRE implementable SCR $F$ is implementable in both LLQREs and strict LLQREs.

Proof. Take any SCR $F$ that is RLLQRE implemented by some $\Gamma$. It is clear that $\Gamma$ implements $F$ in LLQREs. Now let us show that $\Gamma$ implements $F$ in strict LLQREs. Fix any $\theta$. Then by the definition of RLLQRE implementation,

1. for any $a \in F(\theta)$, there exists $m^* \in SL(\Gamma, \theta)$ such that $g(m^*) = a$.

In addition, take any $m' \in SL(\Gamma, \theta)$. As $SL(\Gamma, \theta) \subseteq L(\Gamma, \theta)$, $m' \in L(\Gamma, \theta)$. Then by the definition of RLLQRE implementation,

2. $g(m') \in F(\theta)$ for all $m' \in SL(\Gamma, \theta)$.

The two conditions above mean that $\Gamma$ also implements $F$ in strict LLQREs.

Lemma 3.8 shows that RLLQRE implementation is stronger than strict LLQRE implementation which is equivalent to strict Nash implementation. Hence, the necessary conditions for RLLQRE are at least as strong as the ones for strict Nash implementation. Consequently, QMON and NWA, which are necessary for strict Nash implementation as shown in Cabrales and Serrano (2011), are also necessary for RLLQRE implementation which we state in Theorem 3.9.\(^{12}\)

Theorem 3.9. If SCR $F$ is RLLQRE implementable, then $F$ is quasimonotonic. In addition, if some mechanism in which each agent has at least two messages RLLQRE implements $F$, then $F$ satisfies NWA.\(^{13}\)

\(^{12}\)Earlier versions of this paper prove this theorem without using Cabrales and Serrano (2011) because Theorem 3.9 of this paper and Theorem 3 of Cabrales and Serrano (2011) were proven independently.

\(^{13}\)For NWA to be a necessary condition for strict Nash implementation, one needs a slight restriction on the mechanisms used. Cabrales and Serrano (2011) require that the implementation mechanisms are non-imposing, i.e., for each agent there must exist two messages $m_i$ and $m'_i$ with $g(m_i, m_{-i}) \neq g(m_i, m_{-i})$ for some $m_{-i}$. One can slightly weaken this requirement as stated the theorem.
Proof. This theorem is a consequence of Lemmas 2.4 and 3.8 of this paper and Theorem 3 of Cabrales and Serrano (2011).

Now we turn our attention to the sufficient conditions for RLLQRE implementation. We will derive these conditions for the environments satisfying the following assumption.

**Assumption 3.10.** There exists a player \(i\) and an alternative \(w_i\) such that \(u_i(w_i, \theta) < u_i(a, \theta)\) for all \(\theta \in \Theta\) and \(a \neq w_i\).

This assumption says that there must be at least one agent who has a constant worst alternative in all states. This assumption is rather mild and satisfied in many natural situations such as exchange economies.

The necessary conditions for any double implementation in two equilibrium concepts must be (weakly) more restrictive than the ones for implementation in each individual equilibrium notion. This is, for example, the case for double implementation in strict Nash equilibria and Nash equilibria. However, it turns out that the necessary conditions for RLLQRE implementation, which is equivalent to double implementation in strict LLQREs and LLQREs, is equivalent to the ones for strict LLQRE implementation which we will show in our sufficiency result.

For sufficiency result we must design a mechanism in which each “undesirable” message profile, i.e., one that leads to an implementation of a non-SCR, is not an LLQRE. Strict Nash implementing mechanisms eliminate “undesirable” strict LLQREs but not non-strict ones. We know that non-strict LLQRE is a non-strict Nash equilibrium, but we have very limited information on when non-strict Nash equilibrium is LLQRE (recall that this sometimes depends on the scaling of the utilities). In the Appendix, we identify two very specific properties that no non-strict LLQRE satisfies. To utilize this, we need to design a mechanism in which these two properties are satisfied for all “undesirable” non-strict Nash equilibria. This is the main challenge of the sufficiency result.

**Theorem 3.11.** Suppose \(n \geq 3\) and the environment satisfies Assumption 3.10. If SCR \(F\) satisfies QMON and NWA then \(F\) is both RLLQRE and LLQRE implementable.

We defer the formal proof to the Appendix, but here we present the mechanism used in the proof. To simplify the notation, we assume (without loss of generality) that player 1 has a constant worst alternative \(w_1\). Furthermore, for each \(i, \theta \in \Theta\) and \(a \in F(\theta)\), let \(a_i(a, \theta)\) be an alternative in \(A_i(a, \theta)\). In addition, \(a_1(a, \theta) = w_1\).

In our mechanism \(\Gamma = ((M_i)_{i \in \mathbb{N}}, g)\), the message set of player \(i\) is \(M_i = A \times \Theta \times \Theta \times \{0, 1, 2\}\) and a typical message \(m_i\) is of the form \((a, \theta, \theta, \nu_i)\). The outcome function \(g\) is as follows:

1. Every player sends \(m_i = (a, \theta, \theta, 0)\) where \(a \in F(\theta)\). Then \(g(m) = a\).
2. Every player sends \(m_i = (a, \theta, \theta, 1)\) where \(a \in F(\theta)\) and \(\nu_i = 1\) for at least one player. Then \(g(m) = a_l(a, \theta)\) where \(l\) is the lowest indexed player for whom \(\nu_i \neq 0\).
3. Every player \(i \neq j\) sends \(m_i = (a, \theta, \theta, \nu_i)\) where \(a \in F(\theta)\) and player \(j\) sends message \(m_j = (a_j, \theta_j, \theta_j, \nu_j) \neq (a, \theta, \theta, \nu_j)\). When this case is not covered by rule 2,
(a) $g(m) = a_j$ if $m_j = (a_j, \theta_j, \theta'_j, \nu_j)$, $j \in I(a, \theta, \theta'_j)$, and $a_j \in A_j(a, \theta, \theta'_j)$.

(b) $g(m) = a_j(a, \theta)$ if $m_j$ violates one of the conditions in 3a.

4. Some players send $m_i = (a, \theta, \theta, \nu_i)$ where $a \in F(\theta)$, while at least 2 players send $m_j = (a_j, \theta, \theta_j, \nu_j)$. If $j \in I(a, \theta, \theta_j)$, and $a_j \in A_j(a, \theta, \theta_j)$, then $g(m) = a_l$ where $l$ is the lowest index of the players whose first state differs from the second one in her message. (If all players send $m_j = (a_j, \theta, \theta_j, \nu_j)$, we check whether there exists $a \in F(\theta)$ such that all $j \in I(a, \theta, \theta_j)$ and $a_j \in A_j(a, \theta, \theta_j)$. If such an $a$ exists, then $a_1$ is implemented.)

5. In all other cases,
   (a) $g(m) = a_1$ if $\theta'_1 \neq \theta_1$
   (b) $g(m) = w_1$ if $\theta'_1 = \theta_1$

The above mechanism RLLQRE implements $F$. In our proposed mechanism, each player sends a message consisting of four components: an alternative, two states, and one of 0, 1, or 2. Rule 1 ensures that each SCR alternative in the true state is implemented when every player sends the same message containing this SCR alternative, two true states, and 0. If anyone unilaterally deviates from this message profile, then the outcome function follows either rule 2 or 3 which are both constructed to punish the deviator. This construction is possible thanks to NWA. Therefore, each SCR alternative is delivered by a strict Nash Equilibrium or equivalently by a strict LLQRE (Lemma 2.4).

We need rules 2, 3, 4 and 5 to prove that no LLQRE leads to an implementation of a non-SCR alternative. Here let us only concentrate on the case in which the players send a message profile which follows rule 1 but delivers a non-SCR alternative. In this case QMON and rule 3a guarantee the existence of a player who can unilaterally deviate from the original profile without hurting herself. We have to show that the original profile is not an LLQRE which is proven easily if this player can improve strictly through a unilateral deviation. But a difficult case would arise when there is at least one player who has multiple best responses to the original profile. In this case the key step is to prove that all the optimal unilateral deviations of any player from the original profile lead only to rule 3a. For this step we need rule 2 and we refer interested readers to the formal proof in the Appendix. Then rule 4 ensures that the player with the highest index among those who have multiple best responses to the original profile is indifferent between her best responses as long as every other player plays some best response strategy to the original profile. Now Lemma 3.16 in the Appendix yields that the original message profile is not an LLQRE.

Now let us make several remarks related to our sufficiency result and the mechanism used in it.

**Remark 3.12.** Our mechanism and the one used in Cabrales and Serrano (2011) are significantly different. Their mechanism uses three rules: one similar to our rule 1, another one similar to our 3 and the third one which uses a modulo game. These three rules along with QMON guarantee that any “undesirable” message profile is not a strict Nash equilibrium. For this reason they do not need rules similar to our rules 2 and 4. In our case, we need to further ensure that any “undesirable” profile is also not an LLQRE which is guaranteed thanks to rules 2 and 4. Furthermore, we note that none of our rules use a modulo game. In fact, one can replace rule 3 of the Cabrales-Serrano mechanism with a rule similar to our rule 5 for the sufficiency result of strict Nash implementation.
Remark 3.13. We already mentioned that our mechanism does not use a modulo game or integer game. However, for the proof of our sufficiency result, we restrict our attention only to pure LLQREs, which enables us to dispose of the undesired LLQREs. Jackson (1992) points out the shortcomings of not considering mixed equilibria in the context of Nash implementation. This criticism is not easily addressed in our setting. First, by considering mixed LLQREs, one complicates the analysis significantly. An even more troublesome issue of considering mixed LLQREs is that they are not robust to monotonic (even affine) transformations of the utility functions.\textsuperscript{15}

Remark 3.14. Theorem 3.11 considers the environments in which at least one player has a constant worst alternative (Assumption 3.10). Is the sufficiency result valid without this assumption? If so, rule 5b is not well defined for the mechanism used in the proof. Of course, we can modify rule 5b so that the outcome function implements the alternative that player 1 reports. Then one needs to show that any message profile that follow one of rules 3b and 5b is not LLQRE. This part is particularly challenging if these rules yield an outcome which is a top choice of everyone.

However, we can relax Assumption 3.10 in two important cases\textsuperscript{16}: (1) when each player’s set of most preferred alternatives is singleton in each state or (2) when the SCR satisfies no-veto-power condition (NVP), i.e., whenever an alternative \( a \) is a top choice of at least \( n - 1 \) players in some \( \theta \), \( a \) is an SCR alternative in state \( \theta \). Both of these cases are broad enough to cover some important classes of environments: The former contains the environments in which the players’ preferences are single peaked or strict, while the latter contains economic environments.

Cabrales and Serrano (2011) speculate that QMON and NWA are the relevant conditions for implementation in environments with boundedly rational players because many equilibrium concepts in these environments are closely related to strict Nash equilibria. Their speculation holds in our setting in which the players are mistake-prone even though the mechanism used in this paper is significantly different from the one used in Cabrales and Serrano (2011). Therefore, the two papers complement each other and seem to be consistent with a bigger picture of implementation in environments with boundedly rational players. However, in these environments, perhaps our mechanism is more relevant because it implements in both LLQREs and strict Nash equilibria.

Finally, let us summarize the sufficient conditions for various implementation concepts (considered in this paper) when at least one player has a state-independent worst alternative.

\begin{center}
\begin{tabular}{|c|c|c|c|c|}
\hline
Nash Imp. & Strict Nash Imp. & LLQRE Imp. & Strict LLQRE Imp. & RLLQRE Imp. \\
MON & QMON & QMON & QMON & QMON \\
NVP & NWA & NWA & NWA & NWA \\
\hline
\end{tabular}
\end{center}

\textsuperscript{15}The discussion for mixed LLQREs is the same as the one for non-strict LLQREs.

\textsuperscript{16}The proofs are lengthy, hence we do not provide them in the paper, but they can be provided upon request.
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Appendix

Proof of Lemma 2.4. Let $\pi_{ik}^* = 1$, i.e., $m_{ik}^*$ is the strategy played with probability 1 under $\pi^*$. Define $\Omega_\delta(\pi^*) \equiv \{\pi \in \Delta : \sum_i \sum_k |\pi_{ik} - \pi_{ik}^*| \leq \delta\}$ and let $B_i(\pi)$ be the best response correspondence of player $i$ to $\pi$. Since $\pi^*$ is a strict Nash equilibrium, there exists $\delta$ such that $\pi_{ik}^* = B_i(\pi)$ for any $\pi \in \Omega_\delta(\pi^*)$ and $i \in N$, i.e., $\bar{u}_{ik}^*(\pi) > \bar{u}_{ik}(\pi)$ for all $\pi \in \Omega_\delta(\pi^*)$ and $k \neq k^*$. Consider any strictly increasing sequence $\{\lambda_r\} \uparrow \infty$.

Step 1. The sequence $\{\sigma(\pi, \lambda_r)\}$ absolutely converges to $\pi^*$ on $\Omega_\delta(\pi^*)$.

Proof of Step 1. To prove this step, it suffices to show

$$\max_{\pi \in \Omega_\delta(\pi^*)} \left[ \sum_i \sum_k |\sigma_{ik}(\pi, \lambda_r) - \pi_{ik}^*| \right] \leq \sum_i \max_{\pi \in \Omega_\delta(\pi^*)} \left[ \sum_k |\sigma_{ik}(\pi, \lambda_r) - \pi_{ik}^*| \right] \equiv \xi_r \to 0.$$ 

Observe that

$$\sum_k |\sigma_{ik}(\pi, \lambda_r) - \pi_{ik}^*| = |\sigma_{ik}^*(\pi, \lambda_r) - 1| + \sum_{k \neq k^*} \sigma_{ik}(\pi, \lambda_r) = 2(1 - \sigma_{ik}^*(\pi, \lambda_r)).$$

Plugging the expression above into the expression of $\xi_r$, we obtain

$$\xi_r \equiv \sum_i \max_{\pi \in \Omega_\delta(\pi^*)} \left[ \sum_k |\sigma_{ik}(\pi, \lambda_r) - \pi_{ik}^*| \right] = 2 \left[ n - \sum_i \min_{\pi \in \Omega_\delta(\pi^*)} \sigma_{ik}^*(\pi, \lambda_r) \right].$$

Now showing $\min_{\pi \in \Omega_\delta(\pi^*)} \sigma_{ik}^*(\pi, \lambda_r) \to \lambda_r \to \infty 1$ completes the proof. By the definition of logit quantal response function,

$$\sigma_{ik}^*(\pi, \lambda_r) = \frac{1}{1 + \sum_{k \neq k^*} \exp(-\lambda_r(\bar{u}_{ik}^*(\pi) - \bar{u}_{ik}(\pi))).$$

Consider $\chi \equiv \min_{\pi \in \Omega_\delta(\pi^*)} (\bar{u}_{ik}^*(\pi) - \max_{k \neq k^*} \bar{u}_{ik}(\pi))$. Because the function $\bar{u}_{ik}(\pi) - \max_{k \neq k^*} \bar{u}_{ik}(\pi)$ is continuous, and the set $\Omega_\delta(\pi^*)$ is closed, $\chi$ is well defined and strictly positive. Consequently,

$$\sigma_{ik}^*(\pi, \lambda_r) > \frac{1}{1 + (K_i - 1) \exp(-\lambda_r \chi)).$$

Now observe that the RHS of the inequality above is independent of $\pi$ and converges to 1 as $\{\lambda_r\} \to \infty$. Consequently, $\min_{\pi \in \Omega_\delta(\pi^*)} \sigma_{ik}^*(\pi, \lambda_r)$ converges to 1.
Step 2. There exists a sequence \( \{ \pi_t \} \rightarrow \pi^* \) such that \( \pi_t \in L(G, \lambda_t) \).

Proof of Step 2. As \( \sigma(\pi, \lambda_t) \)s absolutely converge to \( \pi^* \), there must exist \( \bar{\tau} \) such that \( \sigma(\pi, \lambda_t) \in \Omega_3(\pi^*) \) for any \( \pi \in B_3(\pi^*) \) and for all \( \tau > \bar{\tau} \). Consider any \( \tau > \bar{\tau} \). Since \( \Omega_3(\pi^*) \) is convex and compact, and \( \sigma(\pi, \lambda_t) \) is continuous and maps \( \Omega_3(\pi^*) \) to itself, there must exist a fixed point by Brower’s fixed point theorem. Denote this fixed point by \( \pi_\tau \). Since \( \{ \sigma(\pi, \lambda_t) \} \rightarrow \pi^* \) for any \( \pi \in \Omega_3(\pi^*) \), \( \{ \pi_\tau \} \rightarrow \pi^* \). This proves the lemma.

\[ \square \]

Example 3.15. Consider the game in which the strategies and the payoffs are given in Table 3. Then \( (m_{ij})_{i=1,2,3} \) is an LLQRE.

We must show that there exists a sequence of \( \{ p_i \} \rightarrow (m_{ij})_{i=1,2,3} \) such that \( p_i \in L(\lambda_t, \Gamma) \) for some \( \{ \lambda_t \} \rightarrow \infty \). For any \( p \in \{ p_i \} \), the following conditions must be satisfied.

\[ \lambda = \frac{\ln p_{11} - \ln p_{12}}{p_{21}p_{32} + p_{22}p_{31}} = \frac{\ln p_{11} - \ln p_{13}}{p_{21}p_{31} + p_{21}p_{32} + p_{22}p_{31} - 2p_{23}p_{33}} \]

Using the symmetry between the players’ payoffs, we assume that \( p_{11} = p_{21} = p_{31} = \pi_1, p_{12} = p_{22} = p_{32} = \pi_2 \) and \( p_{13} = p_{23} = p_{33} = \pi_3 \). As a result, the above equation reduces to:

\[ 2\pi_1\pi_2\ln \pi_3 + (\pi_1^2 - 2\pi_2^2)\ln \pi_1 - (\pi_1^2 + 2\pi_1\pi_2 - 2\pi_3^2)\ln \pi_2 = 0. \] (6)

Now fix sufficiently small \( \pi_3 > 0 \). We proceed to show that the above equation has a solution.

By definition, \( \pi_2 = 1 - \pi_1 - \pi_3 \). Clearly, the LHS of equation 6 is a continuous function. When \( \pi_1 \rightarrow 0.5, \pi_2 \rightarrow 0.5 - \pi_3 \). Since \( \pi_3 \) is small enough, the term \( 2\pi_1\pi_2\ln \pi_3 \) dominates the LHS of equation 6 as this term approaches \( -\infty \) as \( \pi_3 \rightarrow 0 \) while the other terms are bounded. On the other hand, if \( \pi_1 \rightarrow 1 - \pi_3, \pi_2 \rightarrow 0 \). When \( \pi_2 < \pi_3 \), the LHS of equation 6 is approximately \( -(\pi_1^2 - 2\pi_2^2)\ln \pi_2 \) which converges to \( +\infty \). Now using the intermediate value theorem we obtain that for any small enough \( \pi_3 \), there must exist \( \pi_1(\pi_3) \) and \( \pi_2(\pi_3) \) that satisfy equation 6. Now let us show that \( \pi_1(\pi_3) \rightarrow 1 \) when \( \pi_3 \rightarrow 0 \). Suppose otherwise. Then the term \( 2\pi_1\pi_2\ln \pi_3 \) dominates the LHS of equation 6 as this term approaches \( -\infty \).

Hence, equation 6 is never satisfied. Thus \( \pi_1(\pi_3) \rightarrow 1 \) when \( \pi_3 \rightarrow 0 \). Now set

\[ \lambda(\pi_3) = \frac{\ln \pi_1(\pi_3) - \ln \pi_3}{(\pi_1(\pi_3)) + 2\pi_1(\pi_3)\pi_2(\pi_3) - 2\pi_3^2} \]

Clearly, when \( \pi_3 \rightarrow 0 \), \( \lambda(\pi_3) \rightarrow \infty \). To complete the proof, consider any sequence of \( \pi_3 \)'s converging to 0. Now find the corresponding \( \lambda(\pi_3) \)'s. As argued earlier, the sequence of \( \lambda(\pi_3) \)'s converges to \( \infty \). Now for each \( \lambda(\pi_3) \), everyone playing a strategy in which \( \pi_{11} = p_{21} = p_{31} = \pi_1(\pi_3), p_{12} = p_{22} = p_{32} = \pi_2(\pi_3) \) and \( p_{13} = p_{23} = p_{33} = \pi_3 \) is an LQRE. As argued earlier, this sequence of LQREs must converge to \( (m_{ij})_{i=1,2,3} \).

To prove Theorem 3.11, we identify some properties that non-strict LLQREs never satisfy. For the next lemma, we introduce some new pieces of notation. For each \( \pi \in \Delta \), let \( B_*(\pi) \) be the best response set of player \( i \) to \( \pi \), i.e., \( B_*(\pi) = \{ m_{il} \in M_1 : \bar{u}_{il}(\pi) \leq \bar{u}_{ik}(\pi) \forall m_{ik} \in M_1 \} \). Let \( B(\pi) = \Pi_{i \in N} B_*(\pi) \) and \( B_-(\pi) = \Pi_{j \neq i} B_*(\pi) \).

Lemma 3.16. Let \( \pi^* \) be a pure LLQRE in which each player \( i \) plays \( m_{il}^* \) with probability 1. Then for any player \( i \) there does not exist a strategy \( m_{il} \) such that \( u_i(m_{il}, m_{-i}) \geq u_i(m_{il}^*, m_{-i}) \) for all \( m_{-i} \in B_-(\pi^*) \).
Proof. We prove this lemma by contradiction. Without loss of generality, assume that for player 1, there exists a strategy \( m_{1|1} \) such that \( u_1(m_{1|1}, m_{-1}) \geq u_1(m_{1|1}, m_{-1}) \) for all \( m_{-1} \in B_{-1}(\pi^*) \). First observe that there must be some player \( j \neq 1 \) for whom \( M_1 \setminus B_j(\pi^*) \neq \emptyset \). Otherwise, \( m_{1|1} \) is weakly dominated by \( m_{1|1} \) and a such strategy cannot be played with a probability exceeding 0.5 at any LQRE thanks to the definition of logit quantal response function. This contradicts that \( \pi^* \) is an LQRE. Let \( I^* = \{ i \in N : i \neq 1 \& M_1 \setminus B_i(\pi^*) \neq \emptyset \} \).

As \( \pi^* \in L(G) \), there must exist sequences \( \{ \pi_r \} \rightarrow \pi^* \) and \( \{ \lambda_r \} \rightarrow \infty \) satisfying \( \pi_r \in L(\lambda_r, G) \). Take any \( \lambda \in \{ \lambda_r \} \) and \( \pi \in \{ \pi_r \} \) such that \( \pi \in L(\lambda, G) \). Then \( \pi \) must satisfy:

\[
\frac{\pi_{il^*}}{\pi_{ik}} = \frac{\exp(\lambda u_{il^*}(\pi))}{\exp(\lambda u_{ik}(\pi))} \quad \text{for all} \quad i \in I^* \text{ and } m_{ik} \notin B_i(\pi^*)
\]

Combining the above equations,

\[
(\bar{u}_{l|*}(\pi) - \bar{u}_{l|1}(\pi)) (\ln \pi_{il^*} - \ln \pi_{ik}) = (\bar{u}_{l|*}(\pi) - \bar{u}_{ik}(\pi)) (\ln \pi_{il^*} - \ln \pi_{ij}) \tag{7}
\]

for all \( i \in I^* \) and \( m_{ik} \notin B_i(\pi^*) \). Abusing the notation, let us write \( \pi \rightarrow \pi^* \) instead of \( \{ \pi_r \} \rightarrow \pi^* \). We will reach the desired contradiction if the LHS and RHS of equation 7 converge to different values for some \( i \) and \( m_{ik} \notin B_i(\pi^*) \) when \( \pi \rightarrow \pi^* \).

**RHS:** Since \( \bar{u}_{l|*}(\pi^*) > \bar{u}_{ik}(\pi^*) \) for any player \( i \in I^* \) and \( m_{ik} \notin B_i(\pi^*) \), \( \bar{u}_{il^*}(\pi) > \bar{u}_{ik}(\pi) \) for \( \pi \) close enough to \( \pi^* \). Combining this condition with \( \pi_{il^*} = 1 \) guarantees that the RHS approaches \( \infty \) for all \( m_{ik} \notin B_i(\pi^*) \) when \( \pi \rightarrow \pi^* \).

**LHS:** Now we will show for some \( m_{ik} \notin B_i(\pi^*) \), the LHS does not converge to \( \infty \). As \( \pi_{il^*} \rightarrow 1 \), we concentrate on the term \( (\bar{u}_{l|*}(\pi) - \bar{u}_{l|1}(\pi)) \ln \pi_{ik} \). Let \( \pi(m_{-1}) \) be the probability of the players (except 1) playing \( m_{-1} \). Similarly, \( \pi_i(B_i(\pi^*)) \) is the probability that player \( i \in I^* \) plays a strategy in \( B_i(\pi^*) \). Then

\[
\bar{u}_{l|*}(\pi) - \bar{u}_{l|1}(\pi) = \sum_{m_{-1} \in M_{-1}} (u_1(m_{1|*}, m_{-1}) - u_1(m_{1|1}, m_{-1})) \pi(m_{-1})
\]

\[
\leq \left( 1 - \prod_{j \in I^*} \pi_j(B_j(\pi^*)) \right) \max_{m_{-1} \in M_{-1}} \left| u_1(m_{1|*}, m_{-1}) - u_1(m_{1|1}, m_{-1}) \right|.
\]

To see the inequality above observe that the first term is the probability that player 1’s payoff from \( m_{1|*} \) may exceed the one from \( m_{1|1} \) which happens only if at least one player \( j \in I^* \) plays a strategy outside of \( B_j(\pi^*) \). The second term is the maximal ex-post payoff difference between \( m_{1|*} \) and \( m_{1|1} \). Let \( o \equiv \max_{m_{-1} \in M_{-1}} \left| u_1(m_{1|*}, m_{-1}) - u_1(m_{1|1}, m_{-1}) \right| \). Consequently,

\[
-(\bar{u}_{l|*}(\pi) - \bar{u}_{l|1}(\pi)) \ln \pi_{ik} \leq -o \left( 1 - \prod_{j \neq 1} \pi_j(B_j(\pi^*)) \right) \ln \pi_{ik}
\]

for all \( i \in I^* \) and \( m_{ik} \notin B_i(\pi^*) \). As \( \pi \rightarrow \pi^* \), \( \pi_j(B_j(\pi^*)) \rightarrow 1 \) but \( \ln \pi_{ik} \rightarrow \infty \). Hence, we need to further evaluate the expression above. For all \( i \in I^* \), let \( \pi_{ik} = \max \{ \pi_{ik} : m_{ik} \notin B_i(\pi^*) \} \).
Observe that it must be that \( \pi_{ik} \geq \frac{1 - \pi_i(B_i(\pi^*))}{|M_i| - |B_i(\pi^*)|} \). Hence for \( m_{ik} \),

\[
-o \left(1 - \prod_{j \in I^*} \pi_j(B_j(\pi^*))\right) \ln \pi_{ik} \leq -o \left(1 - \prod_{j \in I^*} \pi_j(B_j(\pi^*))\right) \ln \frac{1 - \pi_i(B_i(\pi^*))}{|M_i| - |B_i(\pi^*)|}.
\]

Let \( \bar{i} \in I^* \) be the player with the lowest \( \pi_i(B_i(\pi^*)) \). Then for player \( \bar{i} \),

\[
-o \left(1 - \prod_{j \in I^*} \pi_j(B_j(\pi^*))\right) \ln \frac{1 - \pi_i(B_i(\pi^*))}{|M_i| - |B_i(\pi^*)|} \leq -o \left(1 - \pi_{i\bar{i}}(B_{i\bar{i}}(\pi^*))\right) \ln \frac{1 - \pi_i(B_i(\pi^*))}{|M_i| - |B_i(\pi^*)|}.
\]

Now applying L’Hospital’s rule and simplifying the terms, we obtain that the expression above converges to 0 as \( \pi_i(B_i(\pi^*)) \to 1 \). This means that for \( \pi \in \{\pi_r\} \), which is close enough to \( \pi^* \), equation 7 is violated, contradicting the supposition \( \pi^* \) is an LLQRE.

**Lemma 3.17.** Let \( s = (s_i)_{i \in N} \) be an LLQRE. Suppose that for each player \( j \in N \), there exists \( t_j \in M_j \) such that \( u_j(s_j, m_{-j}) \neq u_j(t_j, m_{-j}) \) if and only if \( m_{-j} \) satisfies that \( m_i = s_i \) for all \( i < j \) and \( m_i = s_i, t_i \) for all \( i > j \). Then \( u_j(s_j, s_{-j}) > u_j(t_j, s_{-j}) \) for all \( j \in N \).

**Proof.** Consider player \( n \). If \( u_n(s_n, s_{-n}) \leq u_n(t_n, s_{-n}) \), then \( u_n(s_n, m_{-n}) \leq u_n(t_n, m_{-n}) \) for all \( m_{-n} \in M_{-n} \) thanks the condition specified in the lemma. Then Lemma 3.16 yields that \( (s_i)_{i \in N} \) is not an LLQRE. Hence, \( u_n(s_n, s_{-n}) > u_n(t_n, s_{-n}) \). In other words, \( t_n \notin B_n(s) \). Now consider player \( n - 1 \). Let us show that \( u_{n-1}(s_{n-1}, s_{-(n-1)}) > u_{n-1}(t_{n-1}, s_{-(n-1)}) \). Otherwise, \( u_{n-1}(t_{n-1}, m_{-(n-1)}) \geq u_{n-1}(t_{n-1}, m_{-(n-1)}) \) for all \( m_{-(n-1)} \in B_{-(n-1)}(s) \) thanks to the condition specified in the lemma and the fact that \( t_n \notin B_n(s) \). Now Lemma 3.16 yields that \( (s_i)_{i \in N} \) is not an LLQRE. This proves that \( u_{n-1}(s_{n-1}, s_{-(n-1)}) > u_n(t_{n-1}, s_{-(n-1)}) \). The rest of the proof is completed by iteratively applying the same argument sequentially starting from \( n - 2 \) in reverse order.

For the proof of our sufficiency result we need some more pieces of notation. Recall the definitions of \( I(a, \theta, \theta') \) and \( A_i(a, \theta, \theta') \). We decompose \( I(a, \theta, \theta') \) into \( I^1(a, \theta, \theta') \) and \( I^2(a, \theta, \theta') \), and \( A_i(a, \theta, \theta') \) into \( A_i^1(a, \theta, \theta') \) and \( A_i^2(a, \theta, \theta') \). For any player \( i \in I(a, \theta, \theta') \), alternative \( a_i \in A_i(a, \theta, \theta') \) that satisfies the second inequality of condition 4 with strict inequality/equality belongs to \( A_i^1(a, \theta, \theta')/A_i^2(a, \theta, \theta') \). Player \( i \in I(a, \theta, \theta') \) whose \( A_i^1(a, \theta, \theta') \neq \emptyset/A_i^2(a, \theta, \theta') = \emptyset \) belongs to \( I^1(a, \theta, \theta')/I^2(a, \theta, \theta') \).

**Proof of Theorem 3.11.** Consider the mechanism \( \Gamma \) defined in the text.

**Part 1.** For any \( a \in F(\theta) \) and \( \theta \in \Theta \), there exists an LLQRE \( m^* \in L(\Gamma, \theta) \) with \( g(m^*) = a \).

**Proof of Part 1.** Observe that \( a \) is implemented if the players send message profile \( m^* = (a, \theta, \theta, 0)_{i \in N} \). To prove \( m^* \in L(\Gamma, \theta) \), we need to show that \( m^* \in SNE(\Gamma, \theta) \) (Lemma 2.4).

If any player unilaterally deviates from \( m^* \), then the outcome function follows either rule 2 or 3. Obviously, the deviator will be strictly worse off. Therefore, \( m^* \in SNE(\Gamma, \theta) \).

**Part 2.** If \( m' \in L(\Gamma, \theta) \), then \( g(m') \in F(\theta) \).

**Proof of Part 2.** First let us show that \( g(m') \) does not follow any of rules 2-5.

---

17Even though \( \bar{i} \) depends on \( \pi \), we can take the limit when \( \pi_{i\bar{i}}(B_{i\bar{i}}(\pi^*)) \to 1 \) because \( \{\pi_r\} \to \pi^* \).
Suppose that $g(m')$ follows rule 2. Then there exists a player $i$ who sends either $(a, \theta, \theta, 1)$ or $(a, \theta, \theta, 2)$. Observe that the outcome function is constructed so that $i$ is indifferent between the above mentioned two strategies. Now by Lemma 3.16, $m' \notin L(\Gamma, \theta)$ which is a contradiction.

Suppose that $g(m')$ follows rules 3a, 4 or 5a. Then there exists a player $i$ for whom $\theta_i' \neq \theta_i$. Let $\bar{m}_i' = (a_i', \theta_i', \theta_i', \nu_i' \neq \nu_i')$. Then by construction, for any $m_{-i}$ neither $g(m_i', m_{-i})$ nor $g(\bar{m}_i', m_{-i})$ follows rules 1 or 2. In such cases, observe that the outcome function is constructed so that $g(m_i', m_{-i}) = g(\bar{m}_i', m_{-i})$ for all $m_{-i}$. Now by Lemma 3.16, $m' \notin L(\Gamma, \theta)$ which is a contradiction.

Suppose $g(m')$ follows rule 3b. Observe here that the outcome function constructed so that each player $i$ constructed so that for each player $i$ implemented by rule 2. Now observe that player 1 can unilaterally deviate (by appropriately changing her first and second message) from $m'$ and get her top alternative in state $\theta$ implemented. This implies that $a_j(a', \theta')$ is player 1’s top choice in state $\theta$. Consequently, $u_1(a_j(a', \theta')) > u_1(w_1, \theta)$ as $F$ satisfies NWA. Now consider $B_1(m')$ and observe that $(a_j', \theta_j', \theta_j', 0) \notin B_1(m')$ because playing $(a_j', \theta_j', \theta_j', 0)$ against $m'$ yields $w_1$. But then $j$ is indifferent between $m_j'$ and $\bar{m}_j' = (a_j', \theta_j', \theta_j', \nu_j' \neq \nu_j')$ as long as player 1 sends any message $m_1 \in B(m')$. Now by Lemma 3.16, $m'$ is not an LLQRE.

Suppose $g(m')$ follows rule 5b. Then $w_1$ is implemented. But player 1 can obtain her top choice by changing only her first and second messages. This is a contradiction.

Now we are left with the case in which $g(m')$ follows rule 1. Let $m' = (a', \theta', \theta', 0)_{i \in N}$. To simplify the notation let $s' = (a', \theta', \theta', 0)$. As $g(m')$ follows rule 1, $a' \in F(\theta')$. If $a' \in F(\theta)$, then we are done. Hence, for the rest of the proof, we assume that $g(m')$ follows rule 1 and $a' \notin F(\theta)$. We will proceed in several steps to show that $m' \notin L(\Gamma, \theta)$.

Step 1. For each player $i$, $I^1(a', \theta', \theta) = \emptyset$.

Proof of Step 1. If $I^1(a', \theta', \theta) \neq \emptyset$, then some player $i \in I^1(a', \theta', \theta)$ will send a message $(a_i, \theta_i', \theta_i', \nu_i)$ where $a_i \in A_i^1(a', \theta', \theta)$ and get $a_i$ implemented by rule 2a. As $u_i(a_i, \theta) > u_i(a', \theta)$, $(a_i, \theta_i', \theta_i', \nu_i)$ is a profitable deviation. Therefore, $m'$ is not a Nash equilibrium, which implies that $m' \notin L(\Gamma, \theta)$. This is a contradiction.

Step 2. There exists a player $i$ for whom $I^2(a', \theta', \theta) \neq \emptyset$.

Proof of Step 2. Because $a' \notin F(\theta')$ and $F$ is quasimonotonic, there must exist a player $i$ and an alternative $a_i$ such that $u_i(a_i, \theta') < u_i(a', \theta')$, but $u_i(a_i, \theta) \geq u_i(a', \theta)$. As $I^1(a', \theta', \theta) = \emptyset$ from Step 1, $I^2(a', \theta', \theta) \neq \emptyset$.

Step 3. For each player $i$, $u_i(a_i(a', \theta'), \theta) < u_i(a', \theta)$.

Proof of Step 3. For each player $i$, define $t' = (a', \theta', \theta', 1)$. If player $i$ plays $t'$ in response to $m'$, then $a_i(a', \theta')$ is implemented by rule 2. Now observe that the outcome function is constructed so that for each player $j$, $u_j(g(s', m_{-j}), \theta) \neq u_j(g(t', m_{-j}), \theta)$ if and only if $m_{-j}$ satisfies that $m_i = s'$ for all $i < j$ and $m_i = s', t'$ for all $i > j$. Then by Lemma 3.17, $u_i(a_i(a', \theta'), \theta) < u_i(a', \theta)$.
Step 4. If a player $i$ has another best response strategy (in addition to $s'$) to $m'$, then it must be of the form $(a_i, \theta', \theta_2', \nu_i)$ where $a_i \in A_i(a', \theta', \theta_2')$.

Proof of Step 4. Suppose that $\bar{m}_i \neq s'$ is player $i$'s best response to $m'$ and suppose that it is not of the form $(a_i, \theta', \theta_2', \nu_i)$ where $a_i \in A_i(a', \theta', \theta_2')$. If $i$ plays $\bar{m}_i$ in response to $m'$, then by rule 3b, $a_i(a', \theta')$ is implemented. By supposition, $\bar{m}_i$ is one of the best strategies for player $i$ in response to $m'$. Hence, $u_i(a_i(a', \theta'), \theta) = u_i(a_i(a', \theta'))$. But this contradicts the inequality we obtained in Step 3.

Recall that for player $i \in I^2(a', \theta', \theta)$, $B_i(m')$ consists of strategies of the form $(a_i, \theta', \theta_2', \nu_i)$ where $a_i \in A_i(a', \theta', \theta)$ and $a_i \in A_i(a', \theta', \theta_2')$. If each player $i$ sends a message $s_i \in B_i(m')$, then the outcome function follows one of rules 1, 3a and 4. Let $\bar{i}$ be the highest indexed player who has more than 1 best response to $m'$. Now observe that player $\bar{i}$ is indifferent between strategies in $B_{\bar{i}}(m')$ as long as each player $i \in N$ sends a message in $B_i(m')$ thanks to rules 1, 3a and 4. Therefore, by Lemma 3.16 $m' \notin L(\Gamma, \theta)$. This concludes the proof that $m' \notin L(\Gamma, \theta)$ if $g(m') \notin F(\theta)$ and follows rule 1.

References


