Robust Group Strategy-Proofness*

Steven Kivinen † Norovsambuu Tumennasan ‡

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Abstract

Strategy-proofness (SP) is a sought-after property in social choice functions because it ensures that agents have no incentive to misrepresent their private information in the interim stage. On the other hand, group strategy-proofness (GSP) is a notion that is applied to the ex-post stage but not to the interim one. Thus, we propose a new notion of GSP, coined robust group strategy-proofness (RGSP), which ensures that no group benefits by deviating from truth-telling in the interim stage. We show that Vickrey auctions satisfy RGSP. In the problem of allocating indivisible objects, an acyclicity condition on the priorities is both necessary and sufficient for the deferred acceptance rule to satisfy RGSP but is only necessary for the celebrated top trading cycles rule. For the allocation of divisible private goods among agents with single-peaked preferences (Sprumont, 1991), only free disposal, fixed path rules satisfy RGSP within the class of sequential allotment rules.

Keywords: Robust group strategy proofness, Vickrey auctions, Top trading cycles, Deferred acceptance, Acyclic priorities, Free-disposal, Fixed path rules

JEL Classifications: C71, C78, D70, D80

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†Department of Economics, Dalhousie University. Email: kivinen@dal.ca
‡Department of Economics, Dalhousie University. Email: norov@dal.ca
1 Introduction

One of the most desirable properties in social choice functions or rules is strategy-proofness (SP). It guarantees that the participating agents never (strictly) benefit by misrepresenting their private information in any realized state. This definition is equivalent to saying that truthtelling is a weakly dominant strategy in the direct revelation game associated with any strategy-proof rule. Although SP is a demanding notion, many prominent rules are strategy-proof. These include the Deferred Acceptance or Top Trading Cycles rules in the problem of allocating indivisible objects, the Vickrey-Clarke-Groves rule in the provision of public goods, or the Uniform Rule in allotment economies with single-peaked preferences.

For strategy-proof rules, truthtelling is an optimal strategy for each individual when there is asymmetric information. In this sense, SP is a notion that is not only applicable to the ex-post stage but also to the interim stage. Given the prevalence of asymmetric information in socio-economic situations, the importance of strategy-proof rules in practice is deservedly paramount.

Despite its desiderata, SP does not take into account the possibility of group deviations. The notion of group strategy-proofness (GSP) corrects this flaw. Specifically, such rules require that no group is able to misreport their private information and benefit in any realized state.\(^1\) Although this definition is a straightforward adaptation of SP to the possibility of group deviations, unlike its individual version, GSP is not an interim-stage notion. In other words, depending on the informational structure, a group may find it profitable to misreport its private information in the direct revelation games associated with some group strategy-proof rules. We demonstrate this point in the example below.

**Example 1.1** (Motivating Example:). Let us consider the so-called Scarf-Shapley economy with three agents and three indivisible objects. A(nn), B(eth) and C(arol) own object \(a\), \(b\) and \(c\), respectively. The agents have strict preferences, and these objects are allocated according to the celebrated Top Trading Cycles rule, \(f^{TTC}\), which is known to satisfy both SP and GSP.\(^2\) This means that at any state (or equivalently, preference profile), no agent or group of agents can profit by misrepresenting their preferences. We now introduce some informational asymmetry. Suppose that the agents know that Ann and Beth have preferences \(R_A\) and \(R_B\), respectively, represented by the following utility functions:

\[
\begin{align*}
    u_A(c) &= 10, \\
    u_B(c) &= 10, \\
    u_A(b) &= 6, \\
    u_B(a) &= 6, \\
    u_A(a) &= 1, \\
    u_B(b) &= 1.
\end{align*}
\]

\(^1\)See Barberá et al. (2016) and Barberá et al. (2010) which establish sufficient conditions for the equivalence of strategy-proofness and group strategy-proofness.

\(^2\)See Section 3 for the formal definition of the Top Trading Cycles rule.
On the other hand, Carol’s preferences could be either $R_C$ or $\tilde{R}_C$, which are represented by the following utility functions:

\[
\begin{align*}
    u_C(a) &= 10, & u_C(b) &= 6, & u_C(c) &= 1, \\
    \tilde{u}_C(b) &= 10, & \tilde{u}_C(a) &= 6, & \tilde{u}_C(c) &= 1.
\end{align*}
\]

Carol knows her own preferences, but Ann and Beth believe Carol’s preferences are either $R_C$ or $\tilde{R}_C$ with equal probability. Would the agents reveal their preferences truthfully in this case?

Because $f^{TTC}$ is strategy-proof, truthtelling is a dominant strategy for each agent. Hence, in the absence of communication, the agents would reveal their preferences truthfully. Consequently, depending on the Carol’s preferences, their allocation would be

\[
\begin{align*}
    f^{TTC}(R_A, R_B, R_C) &= (c, b, a) \quad \text{and} \quad f^{TTC}(R_A, R_B, \tilde{R}_C) = (a, c, b).
\end{align*}
\]

The expected utilities of Ann and Beth are 5.5.

However, what happens if Ann and Beth can coordinate? In this case, they could misreport their preferences as $\tilde{R}_A$ and $\tilde{R}_B$ which are represented by the following utility functions respectively:

\[
\begin{align*}
    \tilde{u}_A(b) &= 10, & \tilde{u}_A(c) &= 6, & \tilde{u}_A(a) &= 1, \\
    \tilde{u}_B(a) &= 10, & \tilde{u}_B(c) &= 6, & \tilde{u}_B(b) &= 1.
\end{align*}
\]

Then the TTC rule would make the following assignment:

\[
\begin{align*}
    f^{TTC}(\tilde{R}_A, \tilde{R}_B, R_C) = f^{TTC}(\tilde{R}_A, \tilde{R}_B, \tilde{R}_C) = (b, a, c).
\end{align*}
\]

Thus, as a result of the misreport, both Ann and Beth net the expected utility of 6 which is an improvement over truthtelling. Consequently, whether a group deviates from truthtelling could depend on the informational structure even for rules satisfying GSP. However, no individual deviation is profitable regardless of informational structure. ☜

The example above highlights an important point: in the presence of asymmetric information, rules satisfying SP remain non-manipulable by individuals whereas rules satisfying GSP may become manipulable by some groups. This means that something is lost in the standard generalization from SP to GSP. In particular, some group of agents who face uncertainty about the private information of those outside of the group may be able to insure against the risk by jointly misreporting their own preferences. In our opinion, the non-robustness of GSP to asymmetric information – a realistic feature in many economic environments – is a significant drawback. This highlights the need for a
new notion of GSP.

To address this shortcoming, we propose a new notion of GSP, coined robust group strategy-proofness (RGSP). Following the Wilson doctrine and the approach advanced by Bergemann and Morris (2005), our notion of blocking does not assume that the agents have common knowledge on the beliefs of other agents. In particular, we require that a group which agrees to deviate from truthtelling is able to rationalize their collusive agreement. That is, there exists a type set of the coalition members for whom the collusive agreement is preferable to truthtelling for some belief which assigns probability 1 to the event that the blocking coalition members have types in the type set. We then say a rule satisfies RGSP if no coalition can rationalize any of its deviations. Thus, our notion ensures that no group agrees to deviate from truthtelling regardless of asymmetric information or beliefs.

The standard definition of GSP requires that the members of any blocking coalition have degenerate and identical beliefs about the others’ types. This requirement is absent for blocking coalitions under our notion. Consequently, our notion is much more stringent than both individual and group SP. This raises the question of how common robust group strategy-proof rules are. In order to answer this question, we consider some classic settings with well-known rules satisfying GSP.

We first investigate the auction setting in which single, indivisible object is for sale. Here, Vickrey auctions satisfy RGSP. The reason is that only the agent who reports the highest valuation within a blocking coalition may obtain the object. The other members of the coalition neither obtain the object nor pay any positive price. Thus, they will not improve no matter what valuations those who are not in the coalition report. In the problem of allocating indivisible objects among agents with strict preferences, the Deferred Acceptance rule satisfies RGSP when priorities are acyclic, i.e., each object gives its highest priorities to a fixed set of agents (albeit the exact order could vary from object to object) and its lowest ones to the remaining agents in a common order. For the other celebrated rule in this setting – Top Trading Cycles, – the same condition is only necessary. In this sense, the DA outperforms the TTC in terms of RGSP. Finally, we examine the allocation of a divisible resource on the domain of single-peaked preferences. We find the classic uniform rule of Sprumont (1991) violates RGSP. However, for sequential allotment rules, fixed path rules (Moulin, 1999) satisfying free disposal\(^3\) are robust group strategy-proof.

While we motivate RGSP as a result of asymmetric information or the lack of common knowledge on the others’ types, there are other interesting scenarios under which uncertainty arises. For instance, even in the ex-post stage agents may think the others play mixed strategies. In such cases, whether groups stick to truthtelling depends on

\(^3\)Free disposal rules designate an individual who is allocated the left-over resource after satisfying the others. See Section 3.3 for a formal definition.
their beliefs of the others’ play.

The paper proceeds as follows. In the next section we describe our framework and define our notion of group strategy-proofness. In Section 3, we study three different well-known settings and identify some rules satisfying RGSP. Section 4 concludes and the appendix contains proofs.

2 Setup

Let \( N = \{1, \ldots, n\} \) be a finite set of agents where \( n \geq 2 \). Let \( A = A_1 \times \ldots \times A_n \) be the set of alternatives. For \( i \in N \), we call \( A_i \) agent \( i \)'s individual set of alternatives. Let \( X \subseteq A \) be the set of feasible allocations and \( x = (x_1, \ldots, x_n) \in X \) be an allocation. If each feasible allocation \( x \in X \) is constant across agents (i.e., if \( x_i = x_j \) for all \( i, j \in N \)) then the model is a public goods one.

Each agent \( i \) has a preference relation denoted by \( R_i \) on \( A_i \). The notations \( P_i \) and \( I_i \) stand for the strict and the indifference part of \( R_i \), respectively. The set of all possible preferences for \( i \) is \( R_i \), and \( \bar{R} \) stands for \( \times_{i \in N} R_i \). The set of admissible preference profiles is \( R = \times_{i \in N} R_i \). We assume that there are no consumption externalities. Hence, we write \( x R_i y \) if and only if \( x_i \not\equiv y_i \).

Fix any \( R \in R, i \in N \) and \( S \subseteq N \). We use the following conventional notation: \( R_{-i} \equiv (R_j)_{j \neq i}, R \equiv (R_i)_{i \in S}, R_{-S} \equiv (R_i)_{i \notin S}, R \equiv (R_S, R_{-S}), R_{-i} \equiv \times_{j \neq i} R_j \), \( R \equiv R_i \times R_{-i}, R_S \equiv \times_{i \in S} R_i, R_{-S} \equiv \times_{i \notin S} R_i \). If \( S = \{i_1, i_2, \ldots, i_m\} \) then we sometimes write \( R_{i_1 i_2 \cdots i_m} \) for \( R_S \), \( R_{i_1 i_2 \cdots i_m} \) for \( R_S \), \( R_{-i_1 i_2 \cdots i_m} \) for \( R_{-S} \). We will refer to preference profiles as states. Similarly, we use the term type \( R_i \) for agent \( i \) whose preference relation is \( R_i \).

A rule \( f \) is a function that maps each state \( R \in \mathcal{R} \) to a feasible allocation, i.e., \( f : \mathcal{R} \rightarrow X \). The notation \( f_i(R) \) indicates the allocation of agent \( i \) in state \( R \). The planner does not know of the agents’ preferences. Thus, she collects the type reports from the agents and based on this information, determines the allocation according to \( f \).

Let \( \Delta A_i \) be the set of lotteries over \( A_i \). For \( a_i \in \Delta A_i \), we write \( a_i(a_i) \) to denote the probability assigned to alternative \( a_i \). We assume that the agents’ utility functions have the expected utility form. Specifically, for each \( i \), there exists a Bernoulli utility function \( u_i : A_i \times \mathcal{R}_i \rightarrow \mathbb{R} \) such that the expected utility of \( i \) with preferences \( R_i \) from lottery \( a_i \) is\(^4\)

\[
\sum_{a_i \in A_i} a_i(a_i) u_i(a_i, R_i).
\]

As we indicated in the Introduction, the agents in our model know their own type

\(^4\)The following expression assumes that the set of alternatives is countable. In the case of uncountable alternatives, one should use integration with the proper measure.
but not necessarily of the other agents’. In other words, we are interested in the interim stage analysis. Consequently, we assume that each agent $i$ of type $R_i$ has a belief $\beta_i(\cdot|R_i)$ on the other agents’ types. In other words, type $R_i$ assigns probability $\beta_i(R_{-i}|R_i)$ to the event that the others’ types are $R_{-i} \in R_i$. We write $\text{Supp}(\beta_i(R_i))$ for the support of $\beta_i(\cdot|R_i)$. For a given rule $f$, the expected utility of type $R_i$ whose belief is $\beta_i(\cdot|R_i)$ is

$$\sum_{R_{-i} \in R_{-i}} \beta_i(R_{-i}|R_i)u_i(f_i(R), R_i).$$

The formulation above assumes that each agent $i$ reports their types truthfully. It is useful to think that the agents agreed to tell truth in the ex-ante stage but they could renege on the agreement in the interim stage when they find out their own types.

### 2.1 Robust Group Strategy-Proofness

Strategy-proofness has long been a cornerstone of the mechanism design literature. In the direct revelation game associated with a strategy-proof rule, no agent benefits by deviating from truth-telling under any circumstance. Thus, strategy-proof rules provide incredibly strong incentives to report truthfully in the direct revelation game associated with these rules.\(^5\) Before we move on, let us provide the following widely used definition of strategy-proofness.

**Definition 2.1** ((Individual) Strategy-Proofness). A rule $f$ satisfies strategy-proofness (SP) if there does not exist $i \in N$, $R \in R$ and $\tilde{R}_i \in R_i$ such that $f(\tilde{R}_i, R_{-i}) P_i f(R)$.

It is well-known that truth-telling is a (weakly) dominant strategy in the direct revelation game associated with any strategy-proof rule. The advantages of truth-telling for rules satisfying SP become even more apparent when asymmetric information is involved. Here, each agent knows her own preferences or type but not the others’. Thus, the eventual outcome of the game is uncertain even when the agents report their types truthfully. Yet, reporting one’s type truthfully guarantees the best outcome in each possible state. Hence, regardless of one’s belief, truth-telling delivers the highest expected utility to each agent.

Despite its many desirable properties, SP does not guarantee that truth-telling is immune to group deviations. Group strategy-proofness, whose roots are traced to the notions of core and strong Nash equilibrium (Aumann, 1959), takes into account the possibility of collusion among agents. This notion requires that no coalition is able to deviate from truth-telling and unambiguously improve its members. Let us state below the usual definition of group strategy-proofness.\(^6\)

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\(^5\)There has been some recent work arguing that some strategy-proof rules work better in practice than others. See, for instance, Saijo et al. (2007), Li (forthcoming), and Bochet and Tumennasan (2017).

\(^6\)See, for instance, Barberà et al. (2010) or Barberà et al. (2016).
Definition 2.2 (Group Strategy-Proofness). A rule $f$ satisfies group strategy-proofness (GSP) if there does not exist $S, R \in \mathcal{R}$ and $\tilde{R}_S$ such that $f(\tilde{R}_S, R_S) \triangleright f(R)$ for each $i \in S$.

This definition says that in the direct revelation game associated with any rule satisfying GSP, no coalition can improve its members by deviating from truthtelling in any realized state. As one can see from the definition above, GSP is an ex-post notion: after a state is revealed, no group of agents can improve each of its members by jointly misreporting their types. However, our main concern is the interim stage when the agents know their types but not of the others. As we demonstrated in Example 1.1, some groups may improve over truthtelling in the interim stage even if the rule satisfies GSP. On the other hand, as we argued earlier for rules satisfying SP, no agent can profitably deviate from truthtelling regardless of information or beliefs. Hence, the notion of SP is an interim stage notion. We next propose an interim stage notion of GSP.

We assume that the agents agreed to truthtelling in the ex-ante stage. However, in the interim stage in which they have found out their own types, some coalition $S$ is contemplating to deviate from truthful reports. What information they exchange and whether they believe each other could have significant effects on whether the group succeeds in deviating or not. We sidestep these issues and investigate the properties successful deviations must satisfy. To be concrete, suppose that the coalition, after deliberating, agrees to some misreport which we assume is enforceable. The coalition members do not know the types of each other as well as of those not in the coalition. However, these agents must justify their agreement to the collusive report. We follow the Wilson doctrine and use a similar approach to that used in the robust mechanism design literature.\footnote{For instance, see Bergemann and Morris (2005).} Specifically, the agents do not know the other agents’ beliefs but they must rationalize their participation in the collusion. In other words, not all the types of the agents in $S$ improve by colluding. Consistent with this, rational agents must hold only certain beliefs. We supplement this discussion with two examples below:

Example 2.3. The set of agents is $\{1, 2\}$ and the set of (public) alternatives is $\{a, b, c, d\}$. Each agent $i \in \{1, 2\}$ can have three preferences, $R_i, \tilde{R}_i$ and $\hat{R}_i$. The rule $f$ and the preferences are given as follows:

<table>
<thead>
<tr>
<th></th>
<th>$f$</th>
<th>$R_1$</th>
<th>$\tilde{R}_1$</th>
<th>$\hat{R}_1$</th>
<th>$R_2$</th>
<th>$\tilde{R}_2$</th>
<th>$\hat{R}_2$</th>
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<tbody>
<tr>
<td>$R_2$</td>
<td>$a$</td>
<td>$\tilde{R}_2$</td>
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<tr>
<td>$\tilde{R}_1$</td>
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<tr>
<td>$\hat{R}_1$</td>
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<td>$a$</td>
<td>$c$</td>
<td>$b$</td>
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\footnote{For instance, see Bergemann and Morris (2005).}
Rule $f$ satisfies SP. Suppose now agents 1 and 2 agree to report $(\tilde{R}_1, \tilde{R}_2)$. Clearly, agent $\bar{R}_1$ would never agree to $(\tilde{R}_1, \tilde{R}_2)$ because she gets her most preferred object $a$ by reporting her type truthfully. However, she would agree to the same misreport if her type was $R_1$ and if she believes player 2’s type is $R_2$. On the other hand, player 2 would never consent to $(\tilde{R}_1, \tilde{R}_2)$ but would if his type was $\tilde{R}_2$ and if he believes player 1’s type is $R_1$. Thus, even though the players do not know each others’ type, the fact that they agreed to play $(\tilde{R}_1, \tilde{R}_2)$ reveals their types to each other. If other words, player 1 knows that player 2’s type is $\bar{R}_2$ and player 2 knows 1’s type is $R_1$. At this point, player 1 cannot justify her collusion at $(\tilde{R}_1, \tilde{R}_2)$.

**Example 2.4.** The set of agents is $\{1, 2, 3\}$ and the set of (public) alternatives is $\{a, b, c, d, e\}$. Each agent $i \in \{1, 2, 3\}$ can have two preferences, $R_i$ and $\tilde{R}_i$. The rule $f$ and the preferences are given as follows:

<table>
<thead>
<tr>
<th></th>
<th>$f$</th>
<th>$R_1$</th>
<th>$\tilde{R}_1$</th>
<th>$R_2$</th>
<th>$\tilde{R}_2$</th>
<th>$R_3$</th>
<th>$\tilde{R}_3$</th>
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<tr>
<td>$R_3$</td>
<td>$\tilde{R}_3$</td>
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<td>$a$</td>
<td>$c$</td>
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<tr>
<td>$R_2$</td>
<td>$\tilde{R}_2$</td>
<td>$c$</td>
<td>$e$</td>
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<td>$R_1$</td>
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<tr>
<td>$\tilde{R}_1$</td>
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Suppose that agents 1 and 2 agree to play $(\tilde{R}_1, \tilde{R}_2)$. Player 1 has incentives to do so when her preferences are $R_1$ and she believes that players 2 and 3 have types $R_1$ and $R_3$ respectively. Player 2 does the same if his preferences are $R_2$ and he believes players 1 and 3 have types $R_1$ and $\tilde{R}_3$ respectively. The colluding players know each other’s type but not player 3’s who is not in the coalition. Thus, players 1 and 2 both can justify their collusion at $(\tilde{R}_1, \tilde{R}_2)$ even though they do not have the same belief on the type of those who are not in the colluding coalition.

To formalize the discussion earlier, we need some new notations. Let $\mathcal{T}$ be the collection of all subsets of $\mathcal{R}$, i.e., $\mathcal{T} \equiv 2^\mathcal{R}$. The collection $\mathcal{T}$ is a lattice with the natural ordering of set inclusion: $\hat{R}_1 \leq \hat{R}_2$ if $\hat{R}_1 \subseteq \hat{R}_2$. The largest element of $\mathcal{T}$ is $\mathcal{R}$ while the smallest one is $\emptyset$.

Suppose now a member of the blocking coalition $S$ knows that the types of the agents are in $\hat{R}$. She can only justify her agreement to a collusive agreement if it brings a better payoff than truth-telling for some of her beliefs in which the types are in $\hat{R}$. To iteratively eliminate the types that cannot benefit by colluding, for any given $\hat{R}_S \in \mathcal{R}_S$, we define an operator $\xi[\hat{R}_S] : \mathcal{T} \rightarrow \mathcal{T}$ such that $\xi[\hat{R}_S](\mathcal{R}) = \times_{i \in N} \xi_i[\hat{R}_S](\mathcal{R})$ where for each $i \notin S$,

$$
\xi_i[\hat{R}_S](\hat{R}) = \mathcal{R}_i
$$
and for each \( i \in S \):

\[
\xi_i[\tilde{R}_S](\tilde{R}) = \left\{ R_i \in \tilde{R}_i \left| \begin{array}{c}
\sum_{R_{-i} \in \tilde{R}_{-i}} \beta_i(R_{-i}|R_i) u_i(f(\tilde{R}_S, R_{-S}), R_i) \\
\quad > \sum_{R_{-i} \in \tilde{R}_{-i}} \beta_i(R_{-i}|R_i) u_i(f(R), R_i)
\end{array} \right. \text{ for some } \beta_i(-|R_i) \text{ with } \text{Supp}(\beta_i(R_i)) \subseteq \tilde{R}_{-i} \right\}.
\]

We note that the operator \( \xi[\tilde{R}_S] \) is increasing: for any \( \tilde{R}^1 \leq \tilde{R}^2 \Rightarrow \xi[\tilde{R}_S](\tilde{R}^1) \leq \xi[\tilde{R}_S](\tilde{R}^2) \). By Tarski’s fixed point theorem, there exists a maximal fixed point \( \mathcal{R}^{\tilde{R}_S} \) such that \( \xi[\tilde{R}_S](\mathcal{R}^{\tilde{R}_S}) = \mathcal{R}^{\tilde{R}_S} \). We can construct \( \mathcal{R}^{\tilde{R}_S} \) by starting with  \( \mathcal{R} = \) the largest element of \( \mathcal{T} \) – and iteratively applying \( \xi[\tilde{R}_S] \). If \( \mathcal{R} \) is finite, then

\[
\mathcal{R}^{\tilde{R}_S} \equiv \bigcap_{k \geq 1} \xi_k[\tilde{R}_S](\xi[k-1][\tilde{R}_S](\mathcal{R})).
\]

Because \( \mathcal{R} \) is could be infinite, we may need transfinite induction to reach the maximal fixed point.\(^9\) We use the notation \( \mathcal{R}^{\tilde{R}_S,k} \) to denote that

\[
\mathcal{R}^{\tilde{R}_S,k} \equiv \xi[\tilde{R}_S](\xi[k-1][\tilde{R}_S](\mathcal{R})).
\]

Let us investigate the operator \( \xi[\tilde{R}_S] \) closely. As we indicated before, we are assuming that coalition \( S \) agreed to a collusive report \( \tilde{R}_S \). Fix any agent \( i \in S \). Then \( \xi_i[\tilde{R}_S](\mathcal{R}) = \mathcal{R}_i^{\tilde{R}_S,1} \) is the set of \( i \)’s types who for some belief of hers, would benefit if coalition \( S \) plays \( \tilde{R}_S \) instead of truth telling. Those who are not in \( S \) are not aware of \( S \)’s agreement meaning that their types cannot be restricted. In our model, the agents are rational, so the members of \( S \) know that the agents’ type is in \( \mathcal{R}_i^{\tilde{R}_S,1} \). Thus, the members of \( S \) justify the collusive agreement \( \tilde{R}_S \) only by having a belief that the other agents’ types are in \( \mathcal{R}_i^{\tilde{R}_S,1} \). This is feasible for only the types of \( S \) in \( \mathcal{R}_S^{\tilde{R}_S,2} \). We can continue with the same logic and find that if \( S \) is were to play \( \tilde{R}_S \) then the agents in \( S \) must have types in the set \( \mathcal{R}_S^{\tilde{R}_S} \) and believe that the others’ types are in \( \mathcal{R}_{-i}^{\tilde{R}_S} \).

Based on the discussion above, we conclude that if \( \mathcal{R}^{\tilde{R}_S} \neq \emptyset \) then whenever some state in it is realized, \( S \) would agree to the collusion \( \tilde{R}_S \). In addition, the members of \( S \) can justify their agreement. However, if \( \mathcal{R}^{\tilde{R}_S} = \emptyset \) then no type of some agent in \( S \) benefits by playing \( \tilde{R}_S \). Consequently, no type of the agent justifies the collusion \( \tilde{R}_S \). We have already mentioned our approach is just an adaptation of the one in the robust mechanism design literature to our setting. Thus, we also borrow the term “rationalization” from

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\(^8\)Observe here that for any \( i \in S \), \( \xi_i[\tilde{R}_S] \) can be equivalently defined as follows:

\[
\xi_i[\tilde{R}_S](\mathcal{R}) = \left\{ R_i \in \tilde{R}_i \left| \text{ for some } R_{-i} \in \tilde{R}_{-i}, f(\tilde{R}_S, R_{-S}) P_i(f(R)) \right. \right\}.
\]

\(^9\)The potential problem is that the limit of the process described in the text may not be a fixed point. In such cases, we start the above process from the limit as described in ? Lipman (1994).
Definition 2.5 (Rationalization). A coalition $S$ rationalizes a misreport $\tilde{R}_S$ if

$$\mathcal{R}^{\tilde{R}_S} \neq \emptyset.$$

We are now ready to introduce our notion of group strategy proofness.

Definition 2.6 (Robust Group Strategy Proofness). A rule $f : \mathcal{R} \rightarrow X$ satisfies robust group strategy-proofness (RGSP) if there exists no coalition $S$ and its report $\tilde{R}_S$ such that $S$ rationalizes $\tilde{R}_S$.

Let us first note that our notion is stronger than the conventional notions of individual and group SP. SP requires that no agent benefits by deviating from truth telling. Hence, let us focus on the coalitions of size 1. Clearly, $\mathcal{R}^{\tilde{R}_i} = \mathcal{R}^{\tilde{R}_i,1}$ for any $i$ and $\tilde{R}_i$. As we already know each type of agent $i$ weakly prefers truth telling to any other report regardless of beliefs in the direct revelation game associated with any strategy-proof rule. This means that $\mathcal{R}^{\tilde{R}_i,1} = \emptyset$ if the rule is strategy proof. On the other hand, if a rule is not strategy proof, then there exists at least one state $R$ in which some type $R_i$ finds profitable to deviate to another report $\tilde{R}_i$. Consequently, $\mathcal{R}^{\tilde{R}_i}$ must include at least $R$. Hence, $f$ is not robust group strategy proof. This discussion confirms our expectation that if one restricts the size of coalitions to 1 then our notion is equivalent to SP.

Let us now investigate the relation between our notion and GSP. Consider any rule $f$ which is not group strategy proof. This means that there exists a state $R$, a coalition $S$ and a misreport $\tilde{R}_S$ with $u_i(f(\tilde{R}), R_i) > u_i(f(R), R_i)$. Then $R \in \mathcal{R}^{\tilde{R}_S}$ because each member of $S$ prefers the misreport $\tilde{R}_S$ for her belief that assigns probability 1 to $R_i$. Hence, RGSP implies GSP. The opposite is not true. To demonstrate this, let us revisit the motivating example in the Introduction. Consider $(\tilde{R}_A, \tilde{R}_B)$. Observe that $(R_A, R_B) \in \mathcal{R}^{(R_A, R_B)}_{AB}$ because both women would agree to report $(\tilde{R}_A, \tilde{R}_B)$ if they believe that the state is either $(R_A, R_B, R_C)$ or $(R_A, R_B, \tilde{R}_C)$ with equal probability. Consequently, RGSP is much more demanding than GSP.

The definition of GSP is sometimes known as weak GSP. Here, each member of a blocking coalition must strictly improve which is a demanding requirement. For the strong GSP, the blocking coalitions can include those who are indifferent between the status quo and the block. In the definition of RGSP, the blocking coalition must rationalize their blocking. We feel that including those who are indifferent in coalitions contemplating such blocking is rather strong in the context of our notion of GSP.

The notion of RGSP does not rely on the agents’ knowledge of the others’ beliefs. In this sense, we are following the Wilson doctrine which questions the reliance on the

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10 Formally, a rule $f$ satisfies strong GSP if there does not exist $S$, $R \in \mathcal{R}$ and $\tilde{R}_S$ such that $f(\tilde{R}_S, R_{-S}) R_i f(R)$ for each $i \in S$ and $f(\tilde{R}_S, R_{-S}) P_j f(R)$ for some $j \in S$. 

---
common knowledge assumption. As a result, robust group strategy proof rules are highly desirable. A potential pitfall is that the set of rules satisfying RGSP may be very narrow. Clearly, the dictatorship rules in the case of public good economies and the sequential dictatorship rules in the case of private good economies satisfy RGSP. In the next section, we identify some rules satisfying RGSP in widely studied settings.

Beforehand let us first consider some desirable properties of rules in the literature. The first one is efficiency – arguably the most desirable property – which says that no group of agents can be improved without hurting others.

Definition 2.7 (Efficiency). A rule $f$ is efficient if for all $R$, there exists no $y \in X$ such that

\[ y \triangleright_i f(R) \quad \text{for some } i \in N \]

\[ y \triangleright_j f(R) \quad \forall j \in N. \]

The next property is nonbossiness (Satterthwaite and Sonnenschein, 1981), which means one cannot change others’ allocation without affecting her own. This property is vacuously satisfied for public good environments.

Definition 2.8 (Nonbossiness). A rule $f$ is nonbossy if whenever $f_i(\tilde{R}_i, R_{-i}) = f_i(R)$ for some $i \in N$, $\tilde{R}_i$ and $R$ we have that $f(\tilde{R}_i, R_{-i}) = f(R)$.

3 Robust Group Strategy Proof Rules

3.1 Auction

In this model, one indivisible object is given to one of the $n$ agents and monetary transfers are allowed. The set of alternatives for agent $i$ is $A_i = \{0, 1\} \times \mathbb{R}$. The set of feasible allocations is $X = \{(y, t) \in \{0, 1\}^n \times \mathbb{R}^n : \sum_{i \in N} y_i = 1\}$. For each $i \in N$, any preference relation $R_i \in \mathcal{R}_i$ is represented by a utility function of the form $u_i(y_i, t_i, R_i) = y_i v_i(R_i) - t_i$ where $v_i(R_i) \in \mathbb{R}_+$ is valuation of the object for agent $i$ of type $R_i$. The Vickrey rule (also known as the second price auction) $f^v$ specifies that the agent whose index is the lowest among those who have the the highest valuation obtains the object and pays the second highest valuation while the others neither obtain object nor pay any positive amount. That is for any given $R$, $f^v(R) = (x, t)$ is such that (i) $x_i = 1$ if $i = \min\{j : v_i(R_j) = \max_{l \in N} v_l(R_l)\}$ and $x_i = 0$ otherwise, (ii) $t_i = \max_{j \neq i} v_l(R_j)$ if $x_i = 1$, and $t_i = 0$ otherwise.

The Vickrey rule is known to be efficient and group strategy-proof. We will next show that this rule also satisfies RGSP.

Proposition 3.1. The Vickrey rule in auction settings is robust group strategy-proof.
Proof. Suppose that the Vickrey rule does not satisfy RGSP. Then there exist \( S \) and \( \tilde{R}_S \) such that \( R_i^{\tilde{R}_S} \neq \). Let \( i^* \) be the agent with the lowest index among those who has the highest valuation under \( \tilde{R}_S \) in \( S \). This means that if anyone in \( S \) is to win the object it must be \( i^* \) as long as \( S \) reports \( \tilde{R}_S \). Fix \( i \in S \) who is not \( i^* \). We know that for each \( R_i \in R_i^{\tilde{R}_S} \) there must exist a belief \( \beta_i(\cdot | R_i) \) with \( \text{Supp}(\beta_i(\cdot | R_i)) \in R_i^{\tilde{R}_S} \) such that

\[
\sum_{R_{-i} \in R_{-i}} \beta_i(R_{-i} | R_i) u_i(f_i(\tilde{R}_S, R_{-S}), R_i) > \sum_{R_{-i} \in R_{-i}} \beta_i(R_{-i} | R_i) u_i(f_i(R), R_i).
\]

However, we know that no agent \( i \) nets a utility below 0 by reporting her preferences truthfully. Hence, \( S \) consists of \( i^* \). Then \( f^* \) is not strategy-proof which is a contradiction. \( \square \)

3.2 Allocation of Indivisible Goods In Strict Preference Domains

In this subsection, we focus on the allocation of indivisible objects. The set of alternatives for each agent is the same and consist of \( m \geq 2 \) indivisible objects, i.e., \( A_i = O = \{0, o_1, \cdots, o_m\} \), where 0 is the null object. Each object \( o \in O \) can be allocated to up to \( q_o \in \mathbb{Z}_{++} \) agents. For technical convenience, we assume \( q_0 = \infty \). We refer to \( q_o \) as object \( o \)'s quota and the collection of quotas \( q = (q_o)_{o \in O} \) as the quota. An allocation \( x \) is feasible if \( |\{i \in N|x_i = o\}| \leq q_o \) for all \( o \in O \). Agent \( i \)'s set of admissible preferences, \( R_i \), is the set of all possible strict preference relations over \( A_i \). We examine two prominent rules: Gale’s Top Trading Cycles (TTC) and Deferred Acceptance (DA). The former is efficient while the latter is stable.\(^{11}\) In fact, the DA rule always results in the agent optimal stable allocation. To introduce these rules formally, we need to define the priority functions. For each object \( o \in O \), a priority function \( p_o \) is a bijection from \( N \) to \( \{1, \cdots, n\} \). We say agent \( i \) has a higher priority than \( j \) at object \( o \) if \( p_o(i) < p_o(j) \). Let \( p \equiv (p_o)_{o \in O} \).

Top Trading Cycles Mechanism: A rule \( f \) is a TTC rule with respect to priority \( p \) and quota \( q \) if the allocation for a given profile \( R \) is found according the following algorithm. In round 1, the set of available agents is \( N \) and the set of available objects is \( O \).

Round k: Each available agent points to her most preferred available object under \( R \) and each available object points to the agent who has the highest priority among the available agents at the object under \( p \). Among these agents and objects we look for trading cycles where a trading cycle is an ordered set \( \{i^1, o^1, i^2, o^2, \cdots, i^k, o^k\} \) such that for each \( l \in \{1, \cdots, k\}, i^l \) points to object \( o' \) while \( o' \) points to \( i^{l+1} \) where \( i^{k+1} = i^1 \). Each agent in any cycle is matched to the object at which she pointed. The set of available

\(^{11}\)A rule \( f \) is stable if for each \( R \), (i) \( f_i(R) \) \( R_i \) for each \( i \in N \), and (ii) there does not exist \( i \) and \( o \in O \setminus \{0\} \) with \( o \) \( P_i \) \( f_i(R) \) and \( p_o(i) < p_o(j) \) where \( f_j(R) = o \).
agents at round $k + 1$ is modified by eliminating the agents who are matched in round $k$. The set of available objects in round $k + 1$ consist of the objects that have not filled their quotas at round $k$.

The algorithm stops when each agent is matched to an object (possibly the null object).

**Deferred Acceptance Mechanism:** A rule $f$ is a DA rule with respect to priority $p$ if the allocation for a given profile $R$ is found according the following algorithm.

**Round 1:** Each agent $i$ “applies” to her most preferred object. Each object $o$ “holds” up to a maximum of $q_o$ agents with the highest priorities (if there are any) and rejects all others.

**Round $k$:** Each agent $i$ whose application was rejected in the previous round applies to her most preferred object which has not rejected her. Any other object $o$ considers the pool of applicants composed of the current applicants and the agents whom $o$ has been holding from the previous round (if there is any). Each object $o$ “holds” up to a maximum of $q_o$ agents in the pool who have the highest priority and rejects all others while the null object “holds” all its applicants.

The algorithm stops when no applicant is rejected and each object $o$ is assigned to the agents whom it held at the final round.

It is well known that the two rules discussed above satisfy the standard notion of GSP.\textsuperscript{12} Here, we investigate their performance in terms of our notion. First we point out that without proper restrictions on the priorities, neither rule satisfies RGSP.

**Example 3.2.** Let $N = \{1, 2, 3\}$ and suppose that $O = \{a, b, 0\}$. The quota for each object other than the null object is 1. Let the object priorities and the agent preferences be given as follows:

\[
\begin{align*}
    p_a(1) &< p_a(2) < p_a(3), & p_b(1) &< p_b(3) < p_b(2), \\
    a & P_1 b P_1 0, & b & P_1 a P_1 0, \\
    b & P_2 a P_1 0, & b & P_2 0 P_2 a, \\
    a & P_3 b P_1 0, & a & P_3 0 P_3 b.
\end{align*}
\]

If agents 2 and 3 have the respective preferences of $R_2$ and $R_3$ then depending on agent 1’s preferences $f \in \{f^{DA}, f^{TTC}\}$ returns the following allocation:

\[
f(R_1, R_2, R_3) = (a, 0, b) \quad \text{and} \quad f(\tilde{R}_1, R_2, R_3) = (b, a, 0).
\]

\textsuperscript{12}The DA does not satisfy the Strong GSP but the TTC does.
If agents 2 and 3 misreport their preferences as $\tilde{R}_2$ and $\tilde{R}_3$ respectively, then depending on agent 1’s preferences the allocation is as follows:

$$f(R_1, \tilde{R}_2, \tilde{R}_3) = (a, b, 0) \text{ and } f(\tilde{R}_1, \tilde{R}_2, \tilde{R}_3) = (b, a, 0).$$

As one can see, agent 2 prefers $f(\tilde{R}_1, \tilde{R}_2, \tilde{R}_3)$ to $f(R_1, \tilde{R}_2, \tilde{R}_3)$ while agent 3 prefers $f(R_1, \tilde{R}_2, \tilde{R}_3)$ to $f(R)$. In other words, $(R_2, R_3) \in R_{2,3}^{\tilde{R}_2, \tilde{R}_3}$. Thus, both the DA and TTC rules with respect to priority $p$ does not satisfy RGSP.

In the example above each object gives the highest priority to the same agent. Given that there are only three agents, the priorities satisfy both the Ergin and Kesten acyclicity conditions.\textsuperscript{13} The former guarantees that the DA rule is efficient while the latter implies that the TTC rule is stable. However, as the example shows both acyclicity conditions do not guarantee the RGSP of the two rules. We will now strengthen the Ergin acyclicity condition which is both necessary and sufficient for the DA rule to satisfy RGSP.\textsuperscript{14} This condition is also necessary for the TTC rule to satisfy RGSP, but not sufficient. We need one more notation to introduce the condition: for each agent $i$ and object $o$, let $U_o(i)$ denote the set of agents who has better priorities than $i$ at $o$, i.e., $U_o(i) \equiv \{j \in N : p_o(j) < p_o(i)\}$.

**Definition 3.3** (Acyclicity). A priority function and quota pair $(p, q)$ is **acyclic** if there do not exist objects $a, b$ and agents $i, j, k$ satisfying the following two conditions:

(C) $p_a(i) < p_a(j) < p_a(k), \text{ and } p_b(k) < p_b(i) \text{ OR } p_b(k) < p_b(j)$.

(S) There exist disjoints sets $N_a, N_b \subseteq N \setminus \{i, j, k\}$ such that $N_a \subseteq U_a(j), \text{ } N_b \subseteq U_b(i)$, $|N_a| = q_a - 1, \text{ and } |N_b| = q_b - 1$.

Although acyclicity is a technical condition, it turns out to provide an intuitive structure: For any pair of objects $a$ and $b$, an acyclic priority structure divides agents into two classes. Highly ranked agents, specifically those with a priority of at most $q_a + q_b$, must be the same for objects $a$ and $b$. Low ranked agents, specifically those with a priority ranking strictly larger than $q_a + q_b$, must be the same as well. Furthermore, the agents in this second class must have identical ranking across goods $a$ and $b$. The following lemma formalizes this idea.

**Lemma 3.4.** Suppose $(p, q)$ is acyclic. For any $a, b \in O$ and agent $i \in O$,

$$p_a(i) \leq q_a + q_b \iff p_b(i) \leq q_a + q_b$$


\textsuperscript{14}The acyclicity condition we consider in this paper is more demanding than the Ergin acyclicity. However, it is neither stronger nor weaker than Kesten acyclicity.
and

\[ p_a(i) > q_a + q_b \implies p_a(i) = p_b(i). \]

**Proof.** See Appendix. \(\square\)

Our main results of this section relate the DA and TTC rules to acyclicity. The following theorem states that acyclicity is necessary for the DA and TTC rules to satisfy RGSP.

**Theorem 3.5.** If the TTC and DA rules with respect to \((p, q)\) satisfies RGSP then \((p, q)\) is acyclic.

**Proof.** Pick \(f \in \{f^{DA}, f^{TTC}\}\) and suppose that \((p, q)\) is cyclic. By Lemma 3.4 there exist \(a\) and \(b\) such that either (a) \(p_a(\ell) \leq q_a + q_b \iff p_b(\ell) \leq q_a + q_b\) for any \(\ell\) but \(q_a + q_b < p_a(\ell) = p_b(\ell)\) for some \(\ell\) or (b) there exist \(i\) with \(p_a(\ell) = q_a + q_b < p_b(\ell)\). Without loss of generality, assume that \(q_a \leq q_b\).

Case (a). In this case, we can find \(j\) and \(k\) with \(q_a + q_b < p_a(j) < p_a(k)\) and \(q_a + q_b < p_b(k) < p_b(j)\). Let \(i\) be the agent for whom \(p_a(i) = q_a\). By assumption, \(p_b(i) \leq q_a + q_b\). Let \(N_a = \{\ell \in N : p_a(\ell) < q_a\}\). The assumptions of this case allows the construction of \(N_b\) such that \(N_a \cap N_b = \emptyset\), \(|N_b| = q_b - 1\) and for each \(\ell \in N_b\), \(p_a(\ell) \leq q_a + q_b \geq p_b(\ell)\).

Let \(R\) be such that all agents prefer 0 to any object in \(O \setminus \{a, b, 0\}\). Let \(R_{−ijk}\) be such that \(a\) is the most preferred object for each \(\ell \in N_a\), \(b\) for each \(\ell \in N_b\) and 0 for each \(\ell \notin N_a \cup N_b \cup \{i, j, k\}\).

We now consider \(R_{ijk}\) and \(\tilde{R}_{ijk}\) such that

\[
\begin{align*}
\text{a}_P a_P b_P 0, 0 \text{ and } 0 & \text{ a } \tilde{P}_t a \tilde{P}_t b \\
\text{a}_P b_P 0, 0 \text{ and } b & \tilde{P}_j 0 \tilde{P}_j a \\
\text{a}_P b_P k_P 0 & \text{ and } a \tilde{P}_k 0 \tilde{P}_k b.
\end{align*}
\]

Consequently, we have that

\[
\begin{align*}
f_{ijk}(R) &= (a, 0, b) & f_{ij}(R_{jk}, R_{−jk}) &= (a, b, 0) \\
f_{ijk}(\tilde{R}_{ijk}, R_{−i}) &= (0, a, b) & f_{ij}(\tilde{R}_{ijk}, R_{−ijk}) &= (0, b, a).
\end{align*}
\]

Observe that \(j\) prefers \(f(\tilde{R}_{jk}, R_{−jk})\) to \(f(R)\) under \(R_j\). Thus, \((\tilde{R}_{jk}, R_{−jk}) \in R_{\tilde{R}_{ijk}}\). At the same time, \(k\) prefers \(f(\tilde{R}_{ijk}, R_{−ijk})\) to \(f(\tilde{R}_{i}, R_{−i})\) under \(R_k\) meaning that \(R \in R_{\tilde{R}_{ijk}}\). Consequently, in this case, \(f\) does not satisfy RGSP, a contradiction.

Case (b). There must exist \(j\) with \(p_a(j) \leq q_a + q_b < p_b(j)\). There must exist \(k\) with \(p_a(k) > q_a + q_b \geq p_b(k)\). We claim that \(p_a(j) \leq q_a\). Suppose otherwise. Let
$i$ be the agent for whom $p_a(i) = q_a$. By following the exact same steps as in case (a), we reach a contradiction. Hence, $p_a(j) \leq q_a$. By symmetry, $p_b(k) \leq q_b$. Let $N^1 = \{ \ell \in N : p_a(\ell) \leq q_a \& p_b(\ell) > q_a + q_b \}$. We know that $j \in N^1$. Let $n^1 = |N^1|$. Pick any $\ell$ with $q_a < p_a(\ell) \leq q_a + q_b$. We now claim that $p_b(\ell) \leq q_b + q_a - n^1$. Suppose otherwise. This means that $q_b \leq q_b + q_a - n^1 < p_b(\ell) \leq q_a + q_b$. Fix an agent $i \in N^1$. Then we must have that $p_a(i) \leq q_a < p_a(\ell) \leq q_a + q_b < p_a(k)$ and $p_b(k) \leq q_b < q_a + q_a - n^1 < p_b(\ell) \leq q_a + q_b < p_b(i)$. Let $N_a = \{ i' \neq i : p_a(i') \leq q_a \}$. By construction, $|N_a| = q_a - 1$. Clearly, $|U_b(\ell)| \geq q_a + q_b - n^1$. In addition, $N^1 \cap U_b(\ell) = \emptyset$. Note here that at most $a_n - n^1$ agents in $N^1$ can have higher priorities than $\ell$ at $b$. Consequently, $|U_b(\ell) \setminus N_a| \geq q_b + q_a - n^1 - (q_a - n^1) = q_b$. Thus, we can construct $N_b$ such that $|N_b| = q_b - 1$, $k \notin N_b$ and $N_b \subset U_b(\ell) \setminus N_a$. By relabeling $j$ by $\ell$ in the proof of part (a), we reach a contradiction. Consequently, any $i'$ with $q_b + q_a - n^1 < p_b(i') \leq q_a + q_b$ must have $p_a(i') \leq q_a$. Set now $N^2 = N^1 \cup \{ i' \in N : q_b + q_a - n^1 < p_b(i') \leq q_a + q_b \}$. Observe that $|N^2| = 2n^1$. By using the same logic as before, we can show that for each $\ell$ with $q_a < p_a(\ell) \leq q_a + q_b$ must have $p_b(\ell) \leq q_b + q_a - 2n^1$. In turn, any $i'$ with $q_b + q_b - 2n^1 < p_b(i') \leq q_a + q_b$ must have $p_a(i') \leq q_a$. Continue with the same logic, and eventually we find that any $\ell$ with $q_a < p_a(\ell) \leq q_a + q_b$ must have $p_b(\ell) \leq q_b$. However, there are $q_b$ such agents. In addition, as pointed out earlier, $p_b(k) \leq q_b$. Thus, there are at least $q_b + 1$ agents with the top $q_b$ priorities at $b$, a contradiction. 

The result above highlights the importance of acyclicity for truth-telling under DA and TTC: it is impossible for either rule to satisfy RGSP when $(p, q)$ is cyclic. We now turn our attention to the result that acyclicity guarantees the RGSP of the DA. Beforehand, we present the following two lemmas that are instrumental in the proof.

**Lemma 3.6.** Suppose $(p, q)$ is acyclic and let $a$ and $\bar{a}$ be non-null objects with the lowest and highest quotas, respectively. For any $a$ and agent $i$,

$$p_a(i) \leq q_a + q_a \implies p_a(i) \leq q_a + q_a$$

and

$$p_{\bar{a}}(i) > q_a + q_a \implies p_a(i) = p_{\bar{a}}(i).$$

**Proof.** See Appendix.

**Lemma 3.7.** Let $N^* \equiv \{ i \in N : p_a(i) \leq q_a + q_a \}$ and $(p, q)$ be acyclic. Furthermore, let $a$ be the most preferred object of $i$ under some preference profile $R$. If $p_a(i) \leq q_a + q_a$,

(a) $f_i^{DA}(R)$ is at worst the second most preferred object of $i$ under $R$.

(b) $f_i^{DA}(R) = a$ whenever the set

$$N_a(R, i) \equiv \{ j \in N^* : p_a(j) < p_a(i) \& a R_j b, \forall b \in O \}$$
has no more than \( q_a - 1 \) agents.

**Proof.** See Appendix.

We are now ready to present the sufficiency of acyclicity for the DA rule to satisfy RGSP.

**Theorem 3.8.** If \((p, q)\) is acyclic then the DA rule satisfies RGSP.

**Proof.** We will show now that if \((p, q)\) acyclic then the DA rule with respect to \((p, q)\) satisfies RGSP. Suppose otherwise. Then there must exist \(S\) and \(\tilde{R}_S\) such that \(S\) rationalizes \(\tilde{R}_S\).

Clearly, \(|S| \neq 1\); otherwise, we obtain a contradiction with the SP of the DA. We need several steps to complete the proof.

**Claim 1:** If \(S\) can rationalize \(\tilde{R}_S\) then \(S \cap N^*\) can rationalize \(\tilde{R}_{N^* \cap S}\).

**Proof of Claim 1:** Let \(S \cap N^* \neq \emptyset\). For each \(i \in S\), there exists \(R^i \in R_{\tilde{R}_S}^i\) such that

\[
f_{DA}(\tilde{R}_S, R^i_{\neg S}) \leq f_{DA}(R^i).
\]

By Lemma 3.6, for each \(i \in N^*, j \notin N^*\) and \(b \in O\),

\[p_a(i) < p_a(j).\]

It is easy to see that for any \(R\) and \(\tilde{R}_{N^*}\),

\[
f_{DA}(R) = f_{DA}(R_{N^*}, R_{N^*}).
\]

Consequently, for each \(i \in N^* \cap S\) and \(R_{N^*}\),

\[
f_{DA}(\tilde{R}_{S \cap N^*}, R^i_{N^* \setminus S}, R_{N^*}) = f_{DA}(\tilde{R}_S, R^i_{\neg S}) \leq f_{DA}(R^i) = f_{DA}(R^i_{N^*}, R_{N^*}).
\]

Hence, for each \(i \in N^* \cap S\) and \(R_{N^*}\), we have that

\[
f_{DA}(\tilde{R}_{S \cap N^*}, R^i_{N^* \setminus S}, R_{N^*}) \leq f_{DA}(R^i_{N^*}, R_{N^*}).
\]

Consequently, \(S \cap N^*\) can rationalizes \(\tilde{R}_{N^* \cap S}\).

Due to Claim 1, we can assume without loss of generality that \(S \subseteq N^*\).

**Claim 2:** Pick any agent \(i \in S\) and \(R_i\). If there exists \(a\) such that \(p_a(i) \leq q_a + q_b\) and \(aR_i b\) for all \(b\) then \(R_i \notin R_i^{\tilde{R}_S}\).

**Proof of the Claim 2:** We prove the claim by induction. Assume that for any \(j \in S\), \(R_j\)
and $b$ with $p_b(j) \leq \kappa$ where $0 = \kappa < q_0 + q_2$ and $bR_j c$ for all $c$, $R_j \notin \mathcal{R}^*_i$. Pick $i \in S$, $R_i$ and $a$ with $p_a(i) = \kappa + 1$ and $aR_i c$ for all $c$. Suppose $R_i \in \mathcal{R}^*_i$. Then there exists $R \in \mathcal{R}^*_i$ such that

$$f^{DA}(\tilde{R}^S, R_{-S}) P_i f^{DA}(R).$$

If $\kappa \leq q_a - 1$, then $f^{DA}_i(R)$ is $i$’s most preferred object, contradicting the relation above. Hence, let $\kappa \geq q_a$. Let $a_i = f^{DA}_i(R)$ and by Lemma 3.6(b), $a_i$ is the second most preferred object under $R_i$. Thus, $f_i(\tilde{R}^S, R_{-S}) = a$. Consider $N_a(R, i)$ which is defined in the proof of Lemma 3.6(a). The same lemma yields that $|N_a(R, i)| \geq q_a$. By the induction assumption, $N_a(R, i) \cap S = \emptyset$. Then by Lemma 3.6(a), $f_i(\tilde{R}^S, R_{-S}) \neq a$, a contradiction.

By extending the same arguments as above, we obtain the following claim.

**Claim 3:** Let $a^+ \in O \setminus \{a\}$ be an object with $q_a^+ \leq q_b$ for all $b \in O \setminus \{a\}$. Pick any agent $i \in S$ and $R_i$. If $p_a(i) \leq q_a^+ + q_a$ and $aR_i b$ for all $b$ then $R_i \notin \mathcal{R}^*_i$.

**Claim 4:** $S = \emptyset$.

**Proof of Claim 4:** Suppose otherwise. Let $i \in S$ be the agent with the best priority at $a$. Let $N^- \equiv \{j \in N^*_a : p_a(j) < p_a(i)\}$ and $N^+ \equiv \{j \in N^*_a : p_a(j) > p_a(i)\}$. By construction, $N^- \cap S = \emptyset$. By Lemma 3.4, if $p_a(i) \leq q_a^+ + q_a^+$ then $p_a(i) \leq q_a + q_a^+$ for each $a \in O$. This and Claims 2 and 3 imply $i \notin S$. Thus,

$$p_a(i) > q_a + q_a^+.$$

Because $i \in S$, there exists $R \in \mathcal{R}^*_i$ such that

$$f^{DA}(\tilde{R}^S, R_{-S}) P_i f^{DA}(R).$$

Let $b^*$ be the most preferred object of $i$ under $R$. Denote $a^* \equiv f^{DA}_i(\tilde{R}^S, R_{-S})$. Under the DA algorithm at profile $R$, $a^*$ rejects $i$.

Let us partition $O$ into $O^- \equiv \{a \in O : p_a(i) > q_a + q_a^+\}$ and $O^+ \equiv \{a \in O : p_a(i) \leq q_a + q_a^+\}$. By claim 2, $b^* \in O^-$. Pick $a \in O^-$. By Lemma 3.4, $p_a(i) = p_a(i), p_a(j) < p_a(i)$ for $j \in N^-$ and $p_a(j) > p_a(i)$ for $j \in N^+$. Thus, $a \in O^-$ rejects $i$ only in favor of those in $N^-$. We now show that in the first step of the DA algorithm at $R$, at least one object $a \in O^-$ receives $q_a$ applicants in $N^-$. Suppose otherwise. We know that eventually $a^*$ receives $q_{a^*}$ applicants in $O^-$. Thus, at some point someone in $O^-$ must be rejected by some object. Consider the very first step of the DA in which some agent $j \in N^-$ is rejected by some object $b$ which cannot be in $O^-$. By then, $i$ is still held by her most preferred object, say $b^*$. This means $b$ holds at least $q_b$ agents who have priorities better than $q_a + q_a^+$. Consequently, $b^*$ will not reject any applicants who hold its first $q_{b^*} + q_b$
priorities. Because \( b \notin O^+ \), \( p_a(i) \leq q_b + q_a \leq q_b^* + q_a \). Given that \( p_a(i) = p_b^*(i) \) (recall \( b^* \in O^- \)), we obtain that \( i \) gets \( b^* \) under profile \( R \), a contradiction.

Pick any \( b \in O^+ \). By Lemma 3.4, each \( j \in N^* \) must have \( p_b(j) \leq q_b + q_a \). Suppose that someone in \( N^* \) applies to \( b \) in the DA algorithm at a report in which the preferences of \( N^- \) are \( R_{N^-} \). We know that at least one object \( a \in O^- \) receives \( q_a \) applicants in \( N^- \subset N^* \). Everyone in \( N^- \) has one of \( b^* \)'s top \( q_b + q_a \leq q_b + q_a \) priorities. Thus, \( b \) does not reject any applicant in \( N^* \) at any report in which the preferences of \( N^- \) are \( R_{N^-} \).

Consider now \( R \) and \( (\hat{R}_S, R_{-S}) \) in which \( N^- \) reports the same preferences. Thus, those objects in \( O^- \) which receive more applicants from \( N^- \) than their quotas are the same in the step 1 of the DA at the two profiles. Thus, any agent in \( N^* \supset N^- \) who applies to \( b \in O^+ \) is not rejected. Thus, the set of agents in \( N^- \) who are rejected in the first step of the DA is the same at the two profiles. In fact, this is true at any step of the DA. Hence,

\[
f^{{DA}}_{N^-}(R) = f^{{DA}}_{N^-}(\hat{R}_S, R_{-S}).
\]

Clearly, there cannot be any object \( a \in O^- \) such that \( aPb^* \) and \( b^* \) and (strictly) less than \( q_a \) agents in \( N^- \) gets \( a \) under \( R \). Consequently, if \( i \) was assigned to any object \( a \in O^- \), it is because \( b^*Ra \). Hence, \( f^{{DA}}_{i}(R) \neq f^{{DA}}_{i}(\hat{R}_S, R_{-S}) \), a contradiction.

Notice that acyclicity is both sufficient and necessary for DA to satisfy RGSP. However, it is only a necessary condition for TTC to satisfy RGSP. In the following example, we present a case in which the TTC rule violates RGSP under acyclic priorities.

**Example 3.9.** Consider three objects \( \{a, b, c, 0\} \) with \( q_a = q_b = 1 \) and \( q_c = 2 \), and three agents \( \{1, 2, 3\} \). The priority structures are the following:

\[
\begin{align*}
p_a(1) &< p_a(2) < p_a(3), \\
p_b(2) &< p_b(1) < p_b(3), \\
p_c(3) &< p_c(1) < p_c(2)
\end{align*}
\]

Notice that \((p, q)\) is acyclic. We will consider a coalition \( S = \{1, 3\} \) with preferences:

\[
\begin{align*}
b & R_1 a R_1 c R_1 0 \\
b & R_3 a R_3 c R_3 0
\end{align*}
\]

Agent 2’s preferences are one of the following:

\[
\begin{align*}
c & R_2 b R_2 a R_2 0 \\
b & \hat{R}_2 c \hat{R}_2 a \hat{R}_2 0
\end{align*}
\]
Under the TTC, \( f^1_{TTC}(R_1, R_2, R_3) = a, f^2_{TTC}(R_1, R_2, R_3) = b, f^1_{TTC}(R_1, \tilde{R}_2, \tilde{R}_3) = a \) and \( f^3_{TTC}(R_1, \tilde{R}_2, \tilde{R}_3) = c \). Now consider the misreport:

\[
\begin{align*}
&b \tilde{R}_1 \ c \tilde{R}_1 \ a \tilde{R}_1 \ 0 \\
&a \tilde{R}_3 \ c \tilde{R}_3 \ b \tilde{R}_3 \ 0
\end{align*}
\]

The allocations now change. Specifically, \( f^1_{TTC}(\tilde{R}_1, R_2, \tilde{R}_3) = b, f^2_{TTC}(\tilde{R}_1, R_2, \tilde{R}_3) = a, f^1_{TTC}(\tilde{R}_1, \tilde{R}_2, \tilde{R}_3) = c \) and \( f^3_{TTC}(\tilde{R}_1, \tilde{R}_2, \tilde{R}_3) = a \).

Observe that agent 1 prefers \( f^1_{TTC}(\tilde{R}_1, R_2, \tilde{R}_3) = b \) to \( f^1_{TTC}(R_1, R_2, \tilde{R}_3) = a \) while agent 3 prefers \( f^3_{TTC}(\tilde{R}_1, \tilde{R}_2, \tilde{R}_3) = a \) to \( f^3_{TTC}(R_1, \tilde{R}_2, \tilde{R}_3) = c \). Therefore, \( \{1, 3\} \) rationalizes \( \tilde{R}_{1\{1,3\}} \).

The example above along with Theorems 3.5 and 3.8 demonstrate that the DA rule outperforms the TTC rule in terms of RGSP: both rules do not satisfy RGSP if priorities are cyclic but only DA does when priorities are acyclic. This is contrary to the conventional wisdom that out of the two rules, the TTC is more immune to group deviations.15

### 3.3 Division of Finite, Divisible Resources in Single-Peaked Preference Domains

In this section, we concentrate on the setting which was first studied in Sprumont (1991). Here, the planner allocates a divisible, finite stock of a resource among agents with single-peaked preferences. Specifically, \( \Omega > 0 \) is the stock of resource and the set of feasible allocations is \( X = \{x \in \mathbb{R}^n_+: \sum_{i \in N} x_i = \Omega\} \). The set of admissible preferences is the one of single-peaked preferences over \([0, \Omega]\). A preference relation \( R_i \) is single-peaked over \([0, \Omega]\) if there exists \( p(R_i) \subseteq [0, \Omega] \) such that for each \( x_i, y_i \in [0, \Omega] \), the conditions \( y_i < x_i \leq p(R_i) \) or \( p(R_i) \leq x_i < y_i \) imply \( x_i \not\sim y_i \).

A rule that is central in this model is the so-called Uniform rule, \( f^U \), defined for each \( R \in \mathcal{R} \) and each \( i \in N \) as,

\[
f^U_i(R) = \begin{cases} 
\min\{p(R_i), \lambda\} & \text{if } \sum_{i \in N} p(R_i) \geq \Omega \\
\max\{p(R_i), \lambda\} & \text{if } \sum_{i \in N} p(R_i) \leq \Omega
\end{cases}
\]

where \( \lambda \) solves \( \sum_{i \in N} f^U_i(R) = \Omega \).

Sprumont (1991) shows that the only rule which satisfies efficiency, strategy-proofness and equal treatment of equals is the uniform rule.16 Furthermore, this rule is group strategy-proof. However, we show below that the uniform rule does not satisfy RGSP.

---

15This is partly because our notion requires that every member of a deviating coalition to strictly improve while the usual notion considered in the literature requires that at least one member of a deviating coalition improves without hurting the others.

16A rule satisfies the property of equal treatment of equals if \( f_i(R) = f_j(R) \) whenever \( p(R_i) = p(R_j) \).
Example 3.10. Let $N = \{1, 2, 3\}$ and $\Omega = 13$. Let us now show that the uniform rule fails RGSP. Let $p(R_1) = 2$, $p(R_2) = 7$. If $p(R_3) = 3$, then $f^u(R_1, R_2, R_3) = (3, 7, 3)$. On the other hand, if $p(\tilde{R}_3) = 5$, then $f^u(R_1, R_2, \tilde{R}_3) = (2, 6, 5)$. Suppose that agents 1 and 2 report $\tilde{R}_1$ and $\tilde{R}_2$ respectively so that $p(\tilde{R}_1) = 1.5$ and $p(\tilde{R}_2) = 7.5$. Then $f^u(\tilde{R}_1, \tilde{R}_2, R_3) = (2.5, 7.5, 3)$ and $f^u(\tilde{R}_1, \tilde{R}_2, \tilde{R}_3) = (1.5, 6.5, 5)$. Clearly, agent 1 prefers $f^u(\tilde{R}_1, R_2, \tilde{R}_3) = (2.5, 7.5, 3)$ to $f^u(R_1, R_2, R_3) = (3, 7, 3)$ and agent 2 $f^u(\tilde{R}_1, \tilde{R}_2, \tilde{R}_3) = (1.5, 6.5, 5)$ to $f^u(R_1, R_2, \tilde{R}_3) = (2, 6, 5)$. Thus, $(R_1, R_2) \notin R^u(\tilde{R}_1, \tilde{R}_2)$. Hence, $f^u$ violates RGSP.

With the negative result demonstrated above, we widen our search of robust group strategy-proof rules to a larger class of rules. Barberá et al. (1997) identify the class of sequential allotment rules which characterizes the efficient, strategy-proof and replacement monotonic rules.\footnote{See Barberá et al. (1997) for the formal definition of sequential allotment rules.}

**Definition 3.11** (Replacement Monotonicity (Barberá et al., 1997)). A rule $f$ satisfies replacement monotonicity if whenever $f_i(\tilde{R}_i, R_{-i}) \geq f_i(R)$ for some $i$, $R$ and $\tilde{R}_i$,

$$f_j(\tilde{R}_i, R_{-i}) \leq f_j(R), \forall j \neq i.$$  

As pointed out in Barberá et al. (1997), replacement monotonicity implies nonbossiness. In addition, it is well-known that the uniform rule is in the class of sequential allotment rules. Thus, Example 3.10 shows that not every rule in this class satisfies RGSP. To understand why this is the case, let us investigate Example 3.10 closely. There, agent 3’s preference peak determines whether it is an over-demanded ($\sum p_i > \Omega$) or under-demanded ($\sum p_i < \Omega$) problem. This leads to a possibility in which a joint misreport of agents 1 and 2 helps agent 1 in an under-demanded case and agent 2 in an over-demanded case. This suggests that one of the over-demanded and under-demanded cases needs to be ruled out. In a literal sense we cannot achieve this but one can manufacture a similar situation by appointing one agent who is allocated the “leftover” resource after satiating the remaining agents. We call such rules free disposal rules, and only these rules can satisfy RGSP as we will show below.

**Definition 3.12** (Free Disposal Rule). A rule $f$ is a free disposal rule if there exists $i^* \in N$ such that

$$f_{i^*}(R) = \max \left\{ 0, \Omega - \sum_{i \neq i^*} p(R_i) \right\}.$$  

**Theorem 3.13.** If some rule $f$ satisfies RGSP, efficiency and replacement monotonicity then $f$ is a free disposal rule.

**Proof.** See Appendix.
Our next example demonstrates that free disposal is not sufficient for RGSP.

**Example 3.14.** $N = \{1, 2, \cdots, 5\}$ and $\Omega = 20$.

**Case 1:** If $\sum_{i=1}^{4} p(R_i) < 20$, then

$$f_i(R) = \begin{cases} p(R_i) & \text{if } i = \{1, \cdots, 4\} \\ 20 - \sum_{j=1}^{4} p(R_j) & \text{if } i = 5 \end{cases}$$

**Case 2:** If $\sum_{i=1}^{4} p(R_i) \geq 20$, then

$$
\begin{align*}
    f_1(R) &= \min\{p(R_1), 7 + \max\{0, 3 - p(R_3)\} + \max\{0, 10 - p(R_2) - p(R_4)\}\} \\
    f_2(R) &= \min\{p(R_2), 7 + \max\{0, 3 - p(R_4)\} + \max\{0, 10 - p(R_1) - p(R_3)\}\} \\
    f_3(R) &= \min\{p(R_3), \max\{3, 20 - \min\{10, p(R_2) + p(R_4)\} - f_1(R)\}\} \\
    f_4(R) &= \min\{p(R_4), \max\{3, 20 - \min\{10, p(R_1) + p(R_3)\} - f_2(R)\}\} \\
    f_5(R) &= 0
\end{align*}
$$

The rule above can be interpreted as follows: 7 units of resource is earmarked for each of agents 1 and 2, and 3 units are earmarked for each of agents 3 and 4. If one of agents 1 or 3 demands less than her earmarked amount then the other’s earmarked amount increases by the difference. The same is true for agents 2 and 4. If the total demand of agents 1 and 3 is less than 10 units, first increase agent 2’s earmarked amount by the difference and then agent 4’s if 2 demands less than her new earmarked amount. If agents 2 and 4 demand less than 10 then a similar scenario unfolds starting with agent 1’s and then with 2’s earmarked amount. If the total demand of the first four agents is less than 20 then agent 5 is allocated the excess supply.

Clearly, $f$ is a free disposal rule, and in addition, it satisfies SP, efficiency and replacement monotonicity. Unfortunately, it turns out that $f$ does not satisfy RGSP which we prove below.

Suppose that $p(R_1) = 7$, $p(R_2) = 5$, $p(R_3) = p(R_4) = 5$, $p(R_5) = 0$, $p(\tilde{R}_1) = 5$ and $p(\tilde{R}_2) = 7$. Then

$$f(R_1, R_2, R_3, R_4, R_5) = (7, 5, 3, 5, 0) \text{ and } f(\tilde{R}_1, \tilde{R}_2, R_3, R_4, R_5) = (5, 7, 5, 3, 0).$$

If agents 3 and 4 report their preference as $\tilde{R}_3$ and $\tilde{R}_4$ where $p(\tilde{R}_3) = p(\tilde{R}_4) = 4$, then

$$f(R_1, R_2, \tilde{R}_3, \tilde{R}_4, R_5) = (7, 5, 4, 4, 0) \text{ and } f(\tilde{R}_1, \tilde{R}_2, \tilde{R}_3, \tilde{R}_4, R_5) = (5, 7, 4, 4, 0).$$
Clearly,

\[
\begin{align*}
&f(R_1, R_2, \hat{R}_3, \hat{R}_4, R_5) P_3 f(R_1, R_2, R_3, R_4, R_5) \\
&f(\hat{R}_1, \hat{R}_2, \hat{R}_3, \hat{R}_4, R_5) P_4 f(\hat{R}_1, \hat{R}_2, R_3, R_4, R_5).
\end{align*}
\]

Thus, \((R_3, R_4) \in \mathcal{R}_{3,4}(\hat{R}_3, \hat{R}_4)\) yielding that \(f\) does not satisfy RGSP.

The reason that the rule considered in the example above fails RGSP turns out to be the following: the total allocation to agents 1, 2 and 5 is the same for profiles \((R_1, R_2, R_3, R_4, R_5)\) and \((\hat{R}_1, \hat{R}_2, R_3, R_4, R_5)\), but the allocations to agents 3 and 4 differ under the same two profiles even though their preferences remain unchanged. One of them is better off while the other is worse off under one preference profile but the situation reverses under the other. Agents 3 and 4 thus want to avoid such risks which is accomplished through a joint misreport. As we will see next, if a rule is to satisfy RGSP then the scenario we have described here cannot occur. In fact, it turns out that whenever some group’s total allocation increases with a preference change of the group, each remaining agent should receive less of the resource.

**Definition 3.15** (Group Replacement Monotonicity). A rule \(f\) is group replacement monotonic if whenever \(\sum_{i \in S} f_i(\hat{R}_S, R_{-S}) \geq \sum_{i \in S} f_i(R)\) for some \(S \subseteq N\), \(R\) and \(\hat{R}_S\) we have that \(f_j(\hat{R}_S, R_{-S}) \leq f_j(R)\), \(\forall j \in N \setminus S\).

Clearly, group replacement monotonicity is more demanding than replacement monotonicity. The two coincide for free disposal rules if there are four or less agents. In addition, this property can be derived from other well-known properties. For instance, consistency and resource monotonicity, which play a prominent role in the literature, imply group replacement monotonicity. In our setting, both the agent pool and resource stock is fixed. However, to define consistency and resource monotonicity, one considers a collection of allocation problems that differ in the agent pool and resource stock, i.e., a collection of problems like ours. Then consistency and resource monotonicity provide a link between the allocation rules in individual allocation problems. Specifically, consistency says that one has to follow the same rule whenever the same amount of resources is allocated among the same group of agents. For instance, suppose that \(S\) is a subset of the agent pool in two different allocation problems. If the agents in \(S\) report the same preferences in both problems and the allocation rules devote the same total quantity of resources to \(S\), then the members of \(S\) must obtain the same allocation in both problems under consistency. Resource monotonicity says that if the resource stock increases while everything else remains constant, then no agent’s allocation decreases. An obvious consequence of consistency in a specific problem like ours is that if a deviating coalition secures the same total amount of the resource by reporting different preferences then the non-deviators’ allocation does not change. Resource monotonicity in addition to consistency
means that if a deviating coalition increases the allocation of its total resource by misreporting then the non-deviators’ allocation cannot increase. Clearly, this is exactly what group replacement monotonicity requires. Thus, consistency and resource monotonicity imply group replacement monotonicity.

One way to adopt consistency and resource monotonicity to our setting is to see if the rule considered in our problem is justified by a consistent and resource monotonic rule defined for the set of all subproblems derived from ours. To be precise, each subproblem has a resource stock not exceeding \( \Omega \) and a set of agents which is a subset of \( N \). An allocation rule \( \psi \) is a collection of rules for all the possible subproblems. We say a rule \( f \) in our original problem is justified by \( \psi \) if \( f \) coincides with \( \psi \) restricted to our problem. \( ^{18} \)

As we will see later, rules satisfying RGSP must be justified by certain types of allocation rules defined for the domain of subproblems derived from our original problem.

We now show that group replacement monotonicity is a necessary condition.

**Theorem 3.16.** Any rule \( f \) satisfies RGSP, efficiency and replacement monotonicity must satisfy group replacement monotonicity.

**Proof.** See Appendix. \( \square \)

In Footnote 18, we pointed out that group replacement monotonic rules are justified by resource monotonic and consistent allocation rules. Moulin (1999) shows that only the fixed path rules satisfy all of efficiency, strategy-proofness, consistency and resource monotonicity.\(^{19} \) Consequently, the theorem above implies that any robust group strategy-proof rule in the class of sequential allotment rules must be justified by a fixed path rule. We find this surprising as the classes of sequential allotment and fixed path rules are unexpectedly connected for rules satisfying RGSP. We next show that efficiency, strategy-proofness and group replacement monotonicity are sufficient for RGSP.

**Theorem 3.17.** Suppose \( f \) satisfies all of efficiency, strategy-proofness, group replacement monotonicity and free disposal. Then \( f \) satisfies RGSP.

**Proof.** Let \( i^* \) be the agent with \( f_{i^*}(R) = \max\{0, \Omega - \sum_{i \neq i^*} p(R_i)\} \) for all \( R \). Let us denote \( N \setminus \{i^*\} \) by \( N^* \). Suppose that \( f \) does not satisfy RGSP.

**Claim 1:** There exists \( S \) and \( \tilde{R}_S \) such that \( |S| \geq 2, S \subset N^* \) and

\[
R^{\tilde{R}_S} \neq \emptyset.
\]

\(^{18}\)A subproblem is a pair \( \langle S, \omega \rangle \) where \( S \subset N \) and \( \omega \leq \Omega \). In addition, let \( X_S(\omega) = \{x_S \in [0, \omega]^{[S]} : x_i \geq 0 \ \forall i \in S \ \& \ \sum_{i \in S} x_i = \omega\} \). An allocation rule for \( \langle S, \omega \rangle \), \( \psi^{(S, \omega)} \), is a mapping that maps \( R_S \) to \( X_S(\omega) \). An allocation rule \( \psi \) is a collection of allocation rules for all the possible subproblems, i.e., \( \psi \equiv \{\psi^{(S, \omega)}\}_{S \subset N \ \& \ \omega \in [0, \Omega]} \). Then \( \psi \) is consistent if for all \( S, \omega \), and \( T \subset S \), we have \( \psi^{(S, \omega)}(R_S) = \psi^{(S \setminus T, \omega - \sum_{i \in T} \psi_i^{(S, \omega)}(R_S))}(R_{S \setminus T}) \). In addition, it is resource monotonic if each \( \psi^{(S, \omega)} \) is non-decreasing in \( \omega \). A rule \( f : R \to X \) is justified by \( \psi \) if \( f = \psi^{(N, \Omega)} \). It is easy to see that \( f \) is group resource monotonic if and only if \( f \) is justified by some consistent and resource monotonic \( \psi \).

\(^{19}\)See Moulin (1999) for the formal definition of the fixed path rules.
Proof of Claim 1: Because \( f \) is not robust group strategy proof, there must exist \( \bar{S} \) and \( \bar{R}_S \) such that \( \mathcal{R}^{ar{R}_S} \neq \emptyset \). If \( i^* \notin \bar{S} \), then we are done. Suppose \( i^* \in \bar{S} \). Because \( f \) is strategy-proof, \( |\bar{S}| \geq 2 \). For each \( i \in \bar{S} \), fix any \( R_i \in \mathcal{R}^{ar{R}_S}_i \). Then there exists \( R^i_{-i} \in \mathcal{R}^{ar{R}_S}_{-i} \) such that

\[
f(\bar{R}_S, R^i_{-i}) P_i f(R_i, R^i_{-i}).
\]

(1)

Fix \( R^i_{i*} \) with \( p(R^i_{i*}) = 0 \). Because \( f \) is a free disposal rule, \( f(R_i, R^i_{-i}) = f(R_i, R^i_{i*}, R^i_{N^*\setminus\{i\}}) \) and \( f(\bar{R}_S, R^i_{-i}) = f(\bar{R}_S\setminus\{i^*\}, R^i_{*}, R^i_{-S\cup\{i^*\}}) \). Thus, for agent \( i \) of type \( R_i \),

\[
f(\bar{R}_S\setminus\{i^*\}, R^i_{*}, R^i_{-S\cup\{i^*\}}) P_i f(R_i, R^i_{*}, R^i_{N^*\setminus\{i\}}).
\]

The proof is complete once we set \( S = \bar{S} \setminus \{i\} \) and \( \bar{R}_S = \bar{R}_S\setminus\{i^*\} \).

Claim 2: Fix \( R^* \) with \( p(R^*_i) = \Omega \) for all \( i \). There exists \( S \subset N^* \) and \( \bar{R}_S \) such that \( R^*_S \in \mathcal{R}^{ar{R}_S}_S \). In addition, for each \( i \in S \), there exists \( R^i_{-S} \) with

\[
f(\bar{R}_S, R^i_{-S}) P_i f(R^*_S, R^i_{-S}).
\]

Proof of Claim 2: By Claim 1, there exist \( S \subset N^* \) and \( \bar{R}_S \), with \( \mathcal{R}^{ar{R}_S} \neq \emptyset \). Fix any \( i \in S \) and \( R_i \in \mathcal{R}^{ar{R}_S}_i \). Then there exists \( R^i_{-i} \in \mathcal{R}^{ar{R}_S}_{-i} \) such that

\[
f(\bar{R}_S, R^i_{-S}) P_i f(R_i, R^i_{-i}).
\]

(2)

Because \( f \) is a free disposal, efficient rule, for (2) to hold, it must be that \( p(R_i) + \sum_{j \in N^* \setminus \{i\}} p(R^*_j) > \Omega \) and \( p(R_i) > f_i(R_i, R^i_{-i}) \). In addition, (2) and the single-peakedness of preferences yield \( f_i(\bar{R}_S, R^i_{-S}) > f(R_i, R^i_{-i}) \). Furthermore, because \( p(R^*_i) \geq p(R^*_i) > f_i(R_i, R^i_{-i}) \), \( f(R^*_i, R^i_{-i}) = f(R_i, R^i_{-i}) \) (Lemma 4.1.b). As a result,

\[
f(\bar{R}_S, R^i_{-S}) P^*_i f(R^*_S, R^i_{-i}).
\]

This proves that \( R^*_S \in \mathcal{R}^{ar{R}_S}_S \). Consider \( f(R_i, R^*_S, R^i_{-S}) \). By replacing \( R^*_j \) by \( R^*_j \) sequentially for each \( j \in S \setminus \{i\} \) and by using replacement-monotonicity and Lemma 4.1.c, we obtain that

\[
f_i(R_i, R^i_{-i}) \geq f_i(R_i, R^*_S, R^i_{-S}).
\]

We know that \( p(R^*_i) \geq p(R^*_i) > f_i(R_i, R^i_{-i}) \). By Lemma 4.1.b,

\[
f(R^*_S, R^i_{-S}) = f(R_i, R^*_S, R^i_{-S}).
\]

Consequently,

\[
f(\bar{R}_S, R^i_{-S}) P^*_i f(R^*_S, R^i_{-i}).
\]
This completes the proof. We are finally ready to prove the theorem.

Thanks to Claim 2, fix \( S \subseteq \mathbb{N}^* \), \( \tilde{R}_S \) and \( R_i^i - S \) such that for all \( i \in S \),

\[
f(\tilde{R}_S, R_i^i - S) \preceq f(R_i^*, R_i^i - S), \tag{3}
\]

Consider \( \sum_{j \in \mathbb{N} \setminus S} f_j(R_i^*, R_i^i - S) \). Let \( j^* \) be the agent in \( S \) such that \( \forall i \in S \)

\[
\sum_{j \in \mathbb{N} \setminus S} f_j(R_i^*, R_i^i - S) \geq \sum_{j \in \mathbb{N} \setminus S} f_j(R_i^*, R_i^i - S),
\]

By group replacement monotonicity, it must be that

\[
f_j(R_i^*, R_i^i - S) \leq f_j(R_i^*, R_i^i - S), \forall j, i \in S. \tag{4}
\]

Furthermore, we know that

\[
p(\tilde{R}_i) \leq p(R_i^*), \forall i \in S.
\]

and

\[
f_j^*(\tilde{R}_S, R_i^i - S) > f_j^*(R_i^*, R_i^i - S).
\]

Clearly, \( (R_i^*, R_i^i - S) \) is an over-demanded case. However, for the last inequality to hold, there must exist some \( \hat{j} \in S \) \((\hat{j} \neq j^*)\) with

\[
p(\tilde{R}_{\hat{j}}) < f_j(R_i^*, R_i^i - S).
\]

By combining this with (4), we have that

\[
p(\tilde{R}_{\hat{j}}) < f_j(R_i^*, R_i^i - S) \leq f_j(R_i^*, R_i^i - S).
\]

Efficiency (if there is an over-demand) or the fact that \( \hat{j} \neq i^* \) imply that

\[
f_j(\tilde{R}_S, R_{\hat{j}}^i - S) \leq p(\tilde{R}_{\hat{j}}) < f_j(\tilde{R}_S, R_{\hat{j}}^i - S) \leq p(R_{\hat{j}}^*).
\]

By single-peakedness, we have

\[
f(R_i^*, R_i^i - S) \preceq f(\tilde{R}_S, R_i^i - S)
\]

which contradicts (3). Thus, \( f \) must satisfy RGSP.

We now comment on which sequential allotment rules are robust group strategy-proof. As we have shown, such rules satisfy efficiency, strategy-proofness, group replacement monotonicity, and free-disposal. The class of rules that satisfy the first three requirements
consists of the rules that can be justified by fixed path rules of Moulin (1999). By the fourth requirement any rule satisfying RGSP must treat one agent very unfairly in the sense that this agent is allocated 0 or the leftover stock depending on whether the others can be fully satiated. The remaining agents will get their peak whenever it is feasible. If not, \( f \) restricted to these agents could be any fixed path rule, including the celebrated uniform rule.

Finally, let us consider setting in which free disposal is allowed. That is, \( X = \{ x \in \mathbb{R}_+^n : \sum_{i \in N} x_i \leq \Omega \} \). In this environment, any efficient rule must allocate the agents’ peaks whenever the resource is under-demanded. We can incorporate this into the definition of fixed path rules. Then we obtain the following theorem.

**Theorem 3.18.** Suppose that the agents have single-peaked preferences on \([0, \Omega]\) and \( X = \{ x \in \mathbb{R}_+^n : \sum_{i \in N} x_i \leq \Omega \} \). Then any rule \( f \) which is justified by any fixed path rule satisfies RGSP.

The proof of this theorem is a simple consequence of Theorem 3.17. The rule above coincides with the fixed path rule in the overdemanded cases. However, in any underdemanded case, all the agents obtain their preference peaks. Obviously, this rule satisfies each of efficiency, RGSP and group replacement monotonicity. If \( f \) is the uniform rule then it would also satisfy equal treatment of equals.

### 4 Conclusion

In this paper, we proposed a new notion of strategy-proofness (RGSP) which takes into account asymmetric information and uncertainty regarding preference reports. We identified the classes of rules satisfying RGSP in three different settings. First, in single object auction settings, Vickrey auctions are robust group strategy-proof. In the problem of allocating indivisible objects among agents with strict preferences, DA but not necessarily TTC satisfies RGSP when priorities are acyclic. Lastly, in the problem of allocating divisible good among agents with single-peaked preferences, we demonstrate that the uniform rule does not satisfy RGSP. On the other hand, an amendment to the uniform rule that allows free disposal does satisfy RGSP. More generally, only the free disposal rules that are justified by fixed path satisfy RGSP within the class of sequential allotment rules.

To the best of our knowledge, our notion of GSP is the first one in the literature that eliminates group deviations in the presence of informational asymmetries, though it is admittedly a demanding concept. Specifically, any coalition can collectively deviate to some misreport as long as it can be rationalized by each member. Here, the beliefs one can have are very permissive. In this sense, blocking is relatively easy in our setting.
Consequently, the set of rules satisfying RGSP is rather small. Perhaps, one should place additional requirements on the beliefs the members of blocking coalition can have. One natural requirement we have considered is that the blocking coalition members reveal their types to each other truthfully. Clearly, such a modification should enlarge the set of rules satisfying the new version of RGSP. However, this does not happen in the problem of allocating indivisible goods or in the Sprumont setting.\footnote{Formally, a rule }\, f\,\text{ satisfies weak robust group strategy-proofness (WRGSP) if there exists } S, R_S, \tilde{R}_S\text{ and } (R_{n-S})_{i \in S}\text{ such that } u_i(f(R_S, R_{n-S}), R_i) > u_i(f(R_S, \tilde{R}_{n-S}), R_i)\text{ for all } i.\text{ TTC and DA rules with respect to priority } p\text{ are weak robust group strategy-proof (WRGSP) if and only if priorities are acyclic. Within the class of sequential allotment rules, only the group replacement monotonic, free disposal rules satisfy WRGSP. These results can be provided upon request.}

Another open question left is to identify sufficient conditions that guarantee the equivalence of robust group strategy-proof and group strategy-proof rules in general environments. We leave these questions for future research.

**References**


O. Bochet and N. Tumennasan. One truth and a thousand lies: Focal points in mechanism design. working paper, Dalhousie University, 2017.


Thus, \( p_{\text{Consequently}} \).

4.1 Proofs for Section 3.2

Appendix


4.1 Proofs for Section 3.2

Proof of Lemma 3.4. Fix any \( a, b \in O \setminus 0 \) and pick \( i \) with \( p_a(i) \leq q_a + q_b \). In contrast to the lemma, suppose \( p_b(i) > q_a + q_b \). Then there exists \( k \) with \( p_a(k) > q_a + q_b \) but \( p_b(k) \leq q_a + q_b \). First let us show that \( p_a(i) = q_a + q_b \) and \( p_a(k) = q_a + q_b + 1 \). In contrast assume that \( p_a(i) < q_a + q_b \) or \( p_a(k) > q_a + q_b + 1 \). Then let \( j \) be the agent with \( p_a(j) = \max\{p_a(i) + 1, q_a\} \). Observe that \( j \neq k \). Pick now a set \( N_a \subset U_a(j) \setminus \{i, j, k\} \) with \( |N_a| = q_a - 1 \). Consider the set \( U_b(i) \) which satisfies \( |U_b(i)| \geq q_a + q_b \). As a result,

\[
|U_b(i) \setminus \{(i, j, k) \cup N_a\}| \geq q_a + q_b - 2 - (q_a - 1) = q_b - 1.
\]

Thus, we can find \( N_b \subseteq U_b(i) \setminus \{i, j, k\} \) such that \( N_b \cap N_a = \emptyset \) and \( |N_b| = q_b - 1 \). Consequently, \( i, j, k \) and \( a, b \) satisfy both (C) and (S). This contradicts \((p, q)\) is acyclic.

Thus, \( p_a(i) = q_a + q_b \) and \( p_a(k) = q_a + q_b + 1 \). Also, the same proof implies that \( p_b(k) = q_a + q_b \) and \( p_b(i) = q_a + q_b + 1 \). Furthermore, for any \( i', p_a(i') < q_a + q_b \) if and only if \( p_b(i') < q_a + q_b \). Pick any \( k^* \) with \( p_a(k^*) > q_a + q_b + 1 \). Observe that \( p_b(k^*) > q_a + q_b + 1 \). Thus, \( q_a + q_b = p_a(i) < p_a(k) < p_a(k^*) \) and \( q_a + q_b = p_b(k) < p_a(i) < p_a(k^*) \). Furthermore, observe that \( U_a(i) = U_b(k) \) and \( |U_a(k) \setminus \{i, k, k^*\}| = |U_b(i) \setminus \{i, j, k\}| = q_a + q_b - 1 \). We can thus find disjoint sets \( \bar{N}_a \subset U_a(k) \) and \( \bar{N}_b \subset U_b(i) \) such that \( |\bar{N}_a| = q_a - 1 \) and \( |\bar{N}_b| = q_b - 1 \). Then \( i, k, k^*, \bar{N}_a \) and \( \bar{N}_b \) satisfy both (C) and (S). This contradicts the acyclicity of \((p, q)\).

We now show that \( p_a(i) > q_a + q_b \) implies \( p_a(i) = p_b(i) \). Suppose otherwise. Because the first part of this lemma, there must exist \( j \) and \( k \) with \( q_a + q_b < p_a(j) < p_a(k) \) and \( q_a + q_b < p_b(k) < p_b(j) \). Let \( i \) be the agent with \( p_a(i) = q_a + q_b \). By the first part, \( p_a(i) \leq q_a + q_b \). Consider \( i, j, a \) and \( b \), and note that condition (C) is satisfied because \( p_a(i) < p_a(j) < p_a(k) \) and \( p_b(k) < p_b(j) \). In addition, \( p_a(k) > p_a(j) > q_a + q_b \geq p_a(i) \)
Given that \( i \) and replacement monotonic, then

Consider the Sprumont setting. If some rule \( f \) is strategy-proof, efficient and replacement monotonic, then \( f \) satisfies the following conditions.

\( (a) \) \( f \) is peak-only, i.e., for any two profiles \( R \) and \( \tilde{R} \) with \( p(R_i) = p(\tilde{R}_i) \) for all \( i \in N \), \( f(R) = f(\tilde{R}) \).

Proof of Lemma 3.6. Pick any \( a \in O \) and any \( i \) with \( p_a(i) \leq q_a + a \). Because \( q_a \leq q_a \), Lemma 3.4 yields that

\[
p_a(i) \leq q_a + a \implies p_a(i) \leq q_a + q_a \leq q_a + q_a
\]

and

\[
p_a(i) > q_a + q_a \implies p_a(i) = p_a(i) \leq q_a + q_a.
\]

On the other hand, if \( p_a(i) > q_a + q_a \), by Lemma 3.4, \( p_a(i) = p_a(i) \geq q_a + q_a \).

Proof of Lemma 3.7. (a) If \( f_i^{DA}(R) \neq a \), then \( a \) must reject \( i \) at some point. Afterwards, \( a \) holds \( q_a \) agents who have better priorities than \( i \) at \( a \). Let \( i \) apply to her second most preferred object \( b \). By Lemma 3.4, the first \( q_a + q_b \) priorities at both \( a \) and \( b \) belong to the same group of agents. Given that \( q_a \) of these are held by \( a \), \( b \) does not reject any agent who has one of its first \( q_a + q_b \) priorities. Hence, \( i \) is assigned to its second most preferred object.

(b) Suppose that \( |N_a(R, i)| \leq q_a - 1 \). Clearly, \( i \) cannot be rejected by \( a \) in Step 1 of the DA algorithm. Pick any \( j \notin N_a(R, i) \) such that \( p_a(j) < p_a(i) \) and \( j \) prefers some object \( b \) to \( a \). Let \( \tilde{N} = \{ k \in N : p_a(k) \leq q_a + q_b \} \). Clearly, \( p_a(j) < p_a(i) \leq q_a + q_b \), and by Lemma 3.4, \( N^* \) which includes \( i \) and \( j \) holds the first \( q_a + q_b \) priorities at both \( a \) and \( b \). Suppose \( j \) is rejected from \( b \) and then applies to \( a \). In the DA algorithm, \( j \) is rejected only in favor of \( q_b \) agents who in this case are in \( \tilde{N} \). However, \( |\{ k \in N : p_a(k) < p_a \}| \leq q_a + q_b - 1 \). Given that \( q_b \) of these agents are held by \( b \), there can be only \( q_a - 1 \) applicants whose priorities are better than \( i \)'s at \( a \). Hence, \( i \) cannot be rejected by \( a \) when \( j \) applies to \( a \). Given that \( j \) is picked randomly, the proof is complete.

4.2 Proofs for Section 3.3

The following well-known results are used in some of the proofs in Section 3.3.

Lemma 4.1. Consider the Sprumont setting. If some rule \( f \) is strategy-proof, efficient and replacement monotonic, then \( f \) satisfies the following conditions.

\( (a) \) \( f \) is peak-only, i.e., for any two profiles \( R \) and \( \tilde{R} \) with \( p(R_i) = p(\tilde{R}_i) \) for all \( i \in N \), \( f(R) = f(\tilde{R}) \).
(b) If \( p(R_i) \leq f_i(R) \) and \( p(\tilde{R}_i) \leq f_i(R) \) for some \( i \), \( R \) and \( \tilde{R}_i \) then \( f(R) = f(\tilde{R}_i, R_{-i}) \).
Similarly, if \( p(R_i) \geq f_i(R) \) and \( p(\tilde{R}_i) \geq f_i(R) \) for some \( i \), \( R \) and \( \tilde{R}_i \) then \( f(R) = f(\tilde{R}_i, R_{-i}) \).

(c) If \( p(\tilde{R}_i) \geq f_i(R) \geq p(R_i) \) for some \( i \), \( R \) and \( \tilde{R}_i \), then \( f(R_i) \leq f(\tilde{R}_i, R_{-i}) \leq p(\tilde{R}_i) \). Similarly, if \( p(\tilde{R}_i) \leq f_i(R) \leq p(R_i) \) for some \( i \), \( R \) and \( \tilde{R}_i \), then \( f(R_i) \geq f(\tilde{R}_i, R_{-i}) \geq p(\tilde{R}_i) \).

**Proof of Theorem 3.13.** We will prove the theorem in several steps.

**Claim 1:** If there exist \( R_i, R_j, R^i_{-ij} \) and \( R^j_{-ij} \) with \( f_i(R_i, R_j, R^i_{-ij}) > p(R_i) \) and \( f_j(R_i, R_j, R^j_{-ij}) < p(R_j) \), then \( f_j(R_i, R_j, R^j_{-ij}) = 0 \).

**Proof of Claim 1:** Suppose otherwise. Because \( f_i(R_i, R_j, R^i_{-ij}) > p(R_i) \geq 0 \) it must be that \( f_j(R_i, R_j, R^j_{-ij}) < \Omega \). By efficiency,

\[
\begin{align*}
f_i(R_i, R_j, R^j_{-ij}) &\leq p(R_i) < f_i(R_i, R_j, R^j_{-ij}) \\
f_j(R_i, R_j, R^j_{-ij}) &< p(R_j) \leq f_j(R_i, R_j, R^j_{-ij})
\end{align*}
\]

and for any \( k \in N \setminus \{i, j\} \),

\[
\begin{align*}
p(R^k_i) &\leq f_k(R_i, R_j, R^i_{-ij}) \\
f_k(R_i, R_j, R^j_{-ij}) &< p(R^k_i)
\end{align*}
\]

Let \( \tilde{R}^i_{-ij} \) and \( \tilde{R}^j_{-ij} \) be preferences such that \( p(\tilde{R}^i_k) = f_k(R_i, R_j, R^i_{-ij}) \) and \( p(\tilde{R}^j_k) = f_k(R_i, R_j, R^j_{-ij}) \) for each \( k \in N \setminus \{i, j\} \). By Lemma 4.1(b), it must be that \( f(R_i, R_j, \tilde{R}^i_{-ij}) = f(\tilde{R}^i_k, R_j, R^j_{-ij}) \) and \( f(R_i, R_j, \tilde{R}^j_{-ij}) = f(\tilde{R}^i_k, R_j, R^i_{-ij}) \). Fix \( \epsilon > 0 \) such that \( p(R_i) < f_i(R_i, R_j, R^i_{-ij})-\epsilon, 0 < f_j(R_i, R_j, R^j_{-ij})-\epsilon, f_j(R_i, R_j, R^j_{-ij})+\epsilon < p(R_j) \) and \( f_j(R_i, R_j, R^j_{-ij})+\epsilon < \Omega \). Fix \( \tilde{R}_i \) and \( \tilde{R}_j \) with \( p(\tilde{R}_i) = f_i(R_i, R_j, R^i_{-ij})-\epsilon \) and \( p(\tilde{R}_j) = f_j(R_i, R_j, R^j_{-ij})+\epsilon \). Consider now \( f(R_i, \tilde{R}_j, \tilde{R}^i_{-ij}) \). By construction of \( \epsilon \) and \( \tilde{R}_j \), we must have that \( f(R_i) + p(\tilde{R}_j) + \sum_{k \neq i, j} p(\tilde{R}^k_k) < \Omega \). By Lemma 4.1(c), efficiency and replacement monotonicity, we find that \( f_j(R_i, \tilde{R}_j, \tilde{R}^i_{-ij}) = f_i(R_i, R_j, \tilde{R}^i_{-ij})-\epsilon > p(R_i) \), \( f_j(R_i, \tilde{R}_j, \tilde{R}^i_{-ij}) = p(\tilde{R}_j) \) and \( f_k(R_i, \tilde{R}_j, \tilde{R}^i_{-ij}) = p(\tilde{R}^k_k) \) for all \( k \neq i, j \). Because the preferences are single-peaked, \( f(R_i, \tilde{R}_j, \tilde{R}^i_{-ij}) P_i f(R_i, R_j, \tilde{R}^i_{-ij}) \). Finally, let us consider \( (\tilde{R}_i, \tilde{R}_j, \tilde{R}^i_{-ij}) \) which differs from \( (R_i, \tilde{R}_j, \tilde{R}^i_{-ij}) \) only in the preferences of \( i \). Because \( p(\tilde{R}_i) < p(R_i) < f_i(R_i, \tilde{R}_j, \tilde{R}^i_{-ij}) \), by Lemma 4.1(b), it must be that \( f(\tilde{R}_i, \tilde{R}_j, \tilde{R}^i_{-ij}) = f(R_i, \tilde{R}_j, \tilde{R}^i_{-ij}) \). Consequently, we find \( f(R_i, \tilde{R}_j, \tilde{R}^i_{-ij}) P_i f(R_i, R_j, \tilde{R}^i_{-ij}) \). By using the mirror image arguments, we find that \( f(\tilde{R}_i, \tilde{R}_j, \tilde{R}^i_{-ij}) P_j f(R_i, R_j, \tilde{R}^i_{-ij}) \). The last two findings mean that \( (R_i, R_j) \in R^i_{ij} \) contradicting the robust group strategy-proofness of \( f \).

**Claim 2:** For each \( R \) with \( \sum_{j \in N} p(R_j) < \Omega \) there exists a unique agent \( i^R \) with \( f_{i^R}(R) > p(R_{i^R}) \).

**Proof of Claim 2:** Because \( \sum_{i \in N} f_i(R) = \Omega \), efficiency implies the existence of an agent \( j \)
with \( p(R_j) < f_j(R) \). In contrast to the claim suppose that there are more than two such agents. Select any one of them randomly and denote the selected agent by \( i^* \). Fix \( \epsilon > 0 \) so that \( \min_{i \in \{ j : p(R_j) < f_j(R) \}} \{ f_i(R) \} - 4\epsilon > 0 \). Let \( R^1 \) be a preference profile such that

\[
\begin{align*}
p(R^1_i) &= f_{i^*}(R) - 3\epsilon \\
p(R^1_i) &= f_i(R) & \forall i \neq i^*.
\end{align*}
\]

Observe here that

\[
\sum_{i \in N} p(R^1_i) = \Omega - 3\epsilon.
\]

In addition, Lemma 4.1(b) implies that

\[
f(R^1) = f(R).
\]

Consequently,

\[
\begin{align*}
f_{i^*}(R^1) &= f_{i^*}(R) = p(R^1_{i^*}) + 3\epsilon \\
f_i(R^1) &= f_i(R) = p(R^1_i) & \forall i \neq i^*.
\end{align*}
\]

Pick any \( j^* \neq i^* \) with \( f_{j^*}(R) > p(R_{j^*}) \) which is feasible because of the supposition. Fix a preference profile \( R^2 \) such that

\[
\begin{align*}
p(R^2_{j^*}) &= f_{j^*}(R^1) + 2\epsilon \\
p(R^2_i) &= f_i(R^1) & \forall i \neq j^*.
\end{align*}
\]

Observe that \( R^2 \) and \( R^1 \) differ only in agent \( j^* \)'s peak. In addition, \( \sum_{i \in N} p(R^2_i) = \Omega - \epsilon < \Omega \). Now Lemma 4.1(c), efficiency and replacement monotonicity imply that

\[
\begin{align*}
f_{i^*}(R^2) &= f_{i^*}(R^1) - 2\epsilon = f_{i^*}(R) - 2\epsilon = p(R^2_{i^*}) + \epsilon \\
f_{j^*}(R^2) &= f_{j^*}(R^1) + 2\epsilon = f_{j^*}(R) + 2\epsilon = p(R^2_{j^*}) \\
f_i(R^2) &= f_i(R^1) = f_i(R) = p(R^2_i) & \forall i \neq i^*, j^*.
\end{align*}
\]

Pick any agent \( k^* \neq i^*, j^* \). Fix a preference profile \( R^3 \) such that

\[
\begin{align*}
p(R^3_{k^*}) &= f_{k^*}(R^2) + 2\epsilon \\
p(R^3_i) &= f_i(R^2) & \forall i \neq k^*.
\end{align*}
\]

Observe that \( R^3 \) and \( R^2 \) differ only in agent \( k^* \)'s peak. In addition, \( \sum_{i \in N} p(R^3_i) = \Omega + \epsilon > 32 \).
By Lemma 4.1(c), we know that
\[ f_{k^*}(R^3) \geq f_{k^*}(R^2). \]

By replacement monotonicity we know that
\[ f_i(R^3) \leq f_i(R^2) \quad \forall i \neq k^*. \]

Given that \( k^* \)'s allocation at most increases by \( 2\epsilon \), \( i^* \)'s decreases by \( 2\epsilon \) at most. Thus, \( f_{i^*}(R^3) \geq f_{i^*}(R^2) - 2\epsilon = f_{i^*}(R) - 4\epsilon > 0 \) (by construction of \( \epsilon \)). If \( f_i(R^3) < f_i(R^2) = p(R_i^3) \) for some \( i \neq k^* \) then we reach contradiction to Claim 1 by comparing \( R^2 \) and \( R^3 \). Thus, we must have that
\[ f_i(R^3) = f_i(R^2) = p(R_i^3) \quad \forall i \neq k^*, i^*. \] (5)

We could have reached \( R^3 \) from \( R^1 \) by changing \( k^* \)'s preferences first and then \( j^* \)'s. This would give us
\[ f_i(R^3) = p(R_i^3) \quad \forall i \neq j^*, i^*. \]

By combining this with (5), we find that
\[
\begin{align*}
f_i(R^3) &= p(R_i^3) = f_i(R) \quad \forall i \neq i^*, j^*, k^* \\
f_j(R^3) &= p(R_j^3) = f_j(R) + 2\epsilon \\
f_k(R^3) &= p(R_k^3) = f_k(R) + 2\epsilon \\
f_i(R^3) &= p(R_i^3) - \epsilon = f_i(R) - 4\epsilon
\end{align*}
\]

Consider a preference profile \( R^4 \) which satisfies
\[
\begin{align*}
p(R_i^4) &= f_i^*(R) + 2\epsilon > f_{i^*}(R^3) \\
p(R_j^4) &= f_i(R^3) \quad \forall i \neq k^*.
\end{align*}
\]

The profiles \( R^3 \) and \( R^4 \) differ in \( i^* \)'s peak. By Lemma 4.1, we know that
\[ f(R^4) = f(R^3). \]

Consequently, \( f_{i^*}(R^4) < f_{i^*}(R) \) and \( f_j^*(R^4) > f_{i^*}(R) \). These inequalities will reverse if we switch the places of \( i^* \) and \( j^* \) in the sequence of preference changes that reach \( R^4 \) from \( R \). This is the contradiction we are looking for.

**Claim 3:** There exists unique agent \( i^* \) such that whenever \( \sum_{i \in N} p(R_i) < \Omega, \ f_{i^*}(R) = \Omega - \sum_{i \neq i^*} p(R_i) \).

**Proof of Claim 3:** By Claim 2 and efficiency, we know that for each \( R \), there exists \( i^R \),
such that \( f_i(R) = \Omega - \sum_{i \neq i_R} p(R_i) \). To prove the current claim, it suffices to show that for any two profiles \( R \) and \( \tilde{R} \) with \( \sum_{i \in N} p(R_i) < \Omega \) and \( \sum_{i \in N} p(\tilde{R}_i) < \Omega \) we have \( i^R = i^{\tilde{R}} \). Let us partition \( N \setminus \{i^R\} \) into two sets: \( N^- = \{i \in N \setminus \{i^R\} : p(\tilde{R}_i) < p(R_i)\} \) and \( N^+ = \{i \in N \setminus \{i^R\} : p(\tilde{R}_i) \geq p(R_i)\} \). If \( N^- \neq \emptyset \), then fix a random agent \( i \in N^- \). Let us now consider \( f(\tilde{R}_i, R_{-i}) \). Because \( i \)'s preference peak decreased there is still an underdemand. Since \( p(\tilde{R}_i) < p(R_i) = f_i(R) \), Lemma 4.1(c) implies \( f_i(\tilde{R}_i, R_{-i}) \leq f_i(R) \). Then by replacement monotonicity \( f_{i^R}(\tilde{R}_i, R_{-i}) \geq f_{i^R}(R) \). This immediately implies \( i(R_i, R_{-i}) = i^R \). By changing the preferences of those in \( N^- \) sequentially and using the same arguments we find that

\[
i^{(\tilde{R}_i,R_{-i})} = i^R.
\]

We now change \( R_{-N^+} \) to \( \tilde{R}_{-N^+} \) sequentially one agent’s preferences at a time. Observe that along these changes we will always have an underdemand because \( \sum_{i \in N} p(\tilde{R}_i) < \Omega \). Pick any agent \( i \in N^+ \). If \( p(R_i) = p(\tilde{R}_i) \) then by Lemma 4.1(a) and nonbossiness,

\[
f(\tilde{R}_{\{i\} \cup N^-}, R_{\{i\} \cup N^-}) = f(\tilde{R}_{N^-}, R_{\cup N^-}) \& i^{(\tilde{R}_{\{i\} \cup N^-}, R_{\{i\} \cup N^-})} = i^R.
\]

If \( p(R_i) < p(\tilde{R}_i) \), by (b) and (c) of Lemma 4.1, we find

\[
p(R_i) = f_i(R) = f_i(\tilde{R}_{N^-}, R_{\cup N^-}) \leq f_i(\tilde{R}_{\{i\} \cup N^-}, R_{\{i\} \cup N^-}) \leq p(\tilde{R}_i).
\]

Because there is an underdemand, we must have

\[
f_i(\tilde{R}_{\{i\} \cup N^-}, R_{\{i\} \cup N^-}) = p(\tilde{R}_i).
\]

By replacement monotonicity and efficiency, we find that

\[
f_j(\tilde{R}_{\{i\} \cup N^-}, R_{\{i\} \cup N^-}) = p(\tilde{R}_j), \forall j \neq i_R.
\]

Consequently,

\[
i^{(\tilde{R}_{\{i\} \cup N^-}, R_{\{i\} \cup N^-})} = i^R.
\]

By using the same arguments sequentially, we find that

\[
i^{(R_i, \tilde{R}_{-i})} = i^R.
\]

Finally, because \( p(\tilde{R}_j) = f_j(R_{i^R}, \tilde{R}_{-i^R}) \) for all \( j \neq i^R \), we cannot have the case in which \( p(\tilde{R}_{i^R}) > f_{i^R}(R_{i^R}, \tilde{R}_{-i^R}) \) because \( \sum_{j \in N} p(\tilde{R}_j) < \Omega \). Then by Lemma 4.1(b), we have that

\[
f(\tilde{R}) = f(R_{i^R}, \tilde{R}_{-i^R}) \& i^{\tilde{R}} = i^R.
\]
Let \( i^* \) be the agent for whom \( f_{i^*}(R) > p(R_{i^*}) \) for all \( R \) with \( \sum_{i \in N} p(R_i) < \Omega \).

**Claim 4:** For each \( R \) with \( \sum_{j \neq i^*} p(R_j) < \Omega \), \( f_{i^*}(R) = \Omega - \sum_{j \neq i^*} p(R_j) \) and \( f_i(R) = p(R_i) \) for all \( i \neq i^* \).

**Proof of Claim 4:** Claim 3 proves the current claim when \( \sum_{i \in N} p(R_i) < \Omega \). In addition, efficiency yields the claim if \( \sum_{i \in N} p(R_i) = \Omega \). Let us focus on the case in which \( \sum_{i \in N} p_i > \Omega \) but \( \sum_{i \neq i^*} p(R_i) < \Omega \). If \( f_{i^*}(R) < \Omega - \sum_{i \neq i^*} p(R_i) \) then by feasibility, there exists \( i \neq i^* \) with \( f_i(R) > p(R_i) \) which cannot happen when there is an overdemand. Hence, \( f_{i^*}(R) \geq \Omega - \sum_{i \neq i^*} p(R_i) \). Set \( N^* = \{i \in N \setminus \{i^*\} : p(R_i) > 0\} \). If \( N^* = \emptyset \), then \( p(R_{i^*}) > \Omega \) and \( p(R_i) = 0 \) for all \( i \neq i^* \). This case cannot happen in our model. Suppose \( |N^*| \geq 2 \). In contrast to the claim, suppose \( f_{i^*}(R) > \Omega - \sum_{j \neq i^*} p(R_j) \). Fix any \( \epsilon > 0 \) such that \( \epsilon < \min_{i \in N^*} \{p(R_i)\} \) and \( \Omega - \sum_{j \neq i^*} p(R_j) + \epsilon < p(R_{i^*}) \). Let \( \tilde{R}_{i^*} \) be preferences of \( i^* \) such that \( p(\tilde{R}_{i^*}) = \Omega - \sum_{j \neq i^*} p(R_j) + \epsilon < p(R_{i^*}) \). Pick any random agent \( j^* \in N^* \). Consider \( \tilde{R}_{j^*} \) with \( p(\tilde{R}_{j^*}) = 0 \). We now investigate \( \tilde{R}_{j^*}, \tilde{R}_{i^*}, R_{-i^*, j^*} \) and observe that

\[
p(\tilde{R}_{i^*}) + p(\tilde{R}_{j^*}) + \sum_{i \neq i^*, j^*} p(R_i) = p(\tilde{R}_{i^*}) + \sum_{i \neq i^*, j^*} p(R_i) = \Omega - (p(R_{j^*}) - \epsilon) < \Omega.
\]

By Claim 3,

\[
f_{j^*}(\tilde{R}_{j^*}, \tilde{R}_{i^*}, R_{-i^*, j^*}) = 0
\]

\[
f_i(\tilde{R}_{j^*}, \tilde{R}_{i^*}, R_{-i^*, j^*}) = p(R_i) \quad \forall i \neq i^*, j^*
\]

\[
f_{i^*}(\tilde{R}_{j^*}, \tilde{R}_{i^*}, R_{-i^*, j^*}) = \Omega - \sum_{j \neq i^*} f_j(\tilde{R}_{j^*}, \tilde{R}_{i^*}, R_{-i^*, j^*}) > p(\tilde{R}_{i^*})
\]

Consider \( f(\tilde{R}_{i^*}, R_{-i^*}) \). By construction, \( p(\tilde{R}_{i^*}) + \sum_{i \neq i^*} p(R_i) = \Omega + \epsilon \). If \( f_{i^*}(\tilde{R}_{i^*}, R_{-i^*}) < \Omega - \sum_{i \neq i^*} p(R_i) \), then by feasibility, we will have that \( p(R_j) < f_j(\tilde{R}_{i^*}, R_{-i^*}) \) for some \( j \neq i^* \) which cannot happen when there is an overdemand. If \( f_{i^*}(\tilde{R}_{i^*}, R_{-i^*}) > \Omega - \sum_{i \neq i^*} p(R_i) > 0 \), then for some \( j \in N^* \), \( p(R_j) > f_j(\tilde{R}_{i^*}, R_{-i^*}) \). If \( j \neq j^*, i^* \), there would be a contradiction with Claim 1 (compare \( (\tilde{R}_j, \tilde{R}_i, R_{-i^*, j^*}) \) and \( (R_j, \tilde{R}_i, R_{-i^*, j^*}) \)). However, for all \( k^* \neq j^* \), we must have that \( p(R_{k^*}) = f_{k^*}(\tilde{R}_{i^*}, R_{-i^*}) \). Recall that \( j^* \) is randomly picked from \( N^* \) and \( |N^*| \geq 2 \). We could have chosen some other agent in \( N^* \). This would give us that \( p(R_j^*) = f_{j^*}(\tilde{R}_{i^*}, R_{-i^*}) \). Consequently,

\[
f_i(\tilde{R}_{i^*}, R_{-i^*}) = \begin{cases} p(R_i) \quad \forall i \neq i^* \\ \Omega - \sum_{j \neq i^*} p(R_j) < p(\tilde{R}_{i^*}) \end{cases}
\]

Finally, consider \( R \) which differs from \( (\tilde{R}_{i^*}, R_{-i^*}) \) only in \( i^* \)'s preferences. We know that \( p(R_{i^*}) > p(\tilde{R}_{i^*}) > f(\tilde{R}_{i^*}, R_{-i^*}) \). By Lemma 4.1(c), we obtain that \( f(R) = f(\tilde{R}_{i^*}, R_{-i^*}) \),
Proof of Theorem 3.16. By Theorem 3.13, there exists $i^*$ for all $i \neq i^*, j^*$. If the current claim is not true, then it must be that $f_{i^*}(R) > \Omega - p(R_{j^*})$ and $f_{j^*}(R) < p(R_{j^*})$. Pick any agent $i \neq i^*, j^*$ and fix $\epsilon > 0$ such that $\epsilon < \Omega - p(R_{j^*})$. Fix preferences of $i$, $R_i$, such that $p(R_i) = \epsilon$. Consider $(\tilde{R}_i, R_{-i})$. By construction, $p(\tilde{R}_i) + \sum_{j \neq i^*, j} p(R_j) < \Omega$ but $p(\tilde{R}_i) + \sum_{j \neq i} p(R_j) > \Omega$. In addition, two agents, $j^*$ and $i$, have peaks exceeding $0$. Thus, as we showed for the $|N^*| \geq 2$ case, it must be that $p(\tilde{R}_i) = f_i(\tilde{R}_i, R_{-i}) = \epsilon > f_i(R)$ and $p(R_{j^*}) = f_{j^*}(\tilde{R}_i, R_{-i}) > f_{j^*}(R)$. These inequalities are incompatible with replacement monotonicity.

Claim 5. If $\sum_{i \neq i^*} p_i > \Omega$, then $f_{i^*}(R) = 0$.

Proof of Claim 5: Suppose $f_{i^*}(R) > 0$. By efficiency, we know that $p(R_{i^*}) \geq f_{i^*}(R) > 0$. Construct $\tilde{R}$ so that $p(\tilde{R}_{i^*}) = p(R_{i^*})$, $p(\tilde{R}_i) \leq p(R_i)$ for all $i \neq i^*$, and $\sum_{i \neq i^*} p(\tilde{R}_i) = \Omega - \epsilon$ where $\epsilon < f_{i^*}(R) \leq p(R_{i^*})$. Observe that $\sum_{i \in N} p(\tilde{R}_i) > \Omega$. Thus, by Claim 4, $f_i(\tilde{R}_i) = p(\tilde{R}_i)$ for all $i \neq i^*$ and $f_{i^*}(\tilde{R}) = \epsilon$. Now let us change the preferences of those in $N \setminus \{i^*\}$ from $\tilde{R}_{-i}$ to $R_{-i}$ sequentially. By construction, each agent’s peak weakly increases. Thus, whenever we change some agent’s preferences, by (b) and (c) of Lemma 4.1, this agent’s allocation weakly increases. This means that in each step, $i^*$’s allocation weakly decreases by replacement monotonicity. Consequently, $f_{i^*}(R) < \epsilon$ which is a contradiction.

Proof of Theorem 3.16. By Theorem 3.13, there exists $i^*$ such that $f_{i^*}(R) = \max\{0, \Omega - \sum_{j \neq i^*} p(R_j)\}$. Fix any $S$, $R$ and $\tilde{R}_S$ satisfying $\sum_{j \in S} f_j(\tilde{R}_S, R_{-S}) \geq \sum_{j \in S} f_j(R)$.

Claim 1: If $\sum_i p(R_i) \leq \Omega$ then $f_j(\tilde{R}_S, R_{-S}) \leq f_j(R)$, $\forall j \in N \setminus S$.

Proof of Claim 1: Because $f$ is a free disposal rule, efficiency implies that $f_i(R) = p(R_i)$ for all $i \neq i^*$ and $f_{i^*}(R) = \Omega - \sum_{i \neq i^*} f_i(R)$. If $\sum_{i \in S \setminus \{i^*\}} p(\tilde{R}_i) + \sum_{i \in S \setminus \{i^*\}} p(R_i) \leq \Omega$, again we have $p(\tilde{R}_i) = f_i(\tilde{R}_S, R_{-S})$, for all $i \in S \setminus \{i^*\}$, $p(R_j) = f_j(\tilde{R}_S, R_{-S})$ for all $j \in N \setminus (S \cup \{i^*\})$, and $f_i(\tilde{R}_S, R_{-S}) = \Omega - \sum_{i \neq i^*} f_i(\tilde{R}_S, R_{-S})$. If $i^* \in S$, we have that $f_j(R) = f_j(\tilde{R}_S, R_{-S})$ for each $j \notin N \setminus S$ which is what we are looking for. If $i^* \notin S$, then each $j \in N \setminus S$ other than $i^*$ gets their peak under $f(\tilde{R}_S, R_{-S})$. But because $f_j(R) = f_j(\tilde{R}_S, R_{-S})$ for each $j \in N \setminus S$ and $\sum_{i \in S} f_i(\tilde{R}_S, R_{-S}) \geq \sum_{i \in S} f_i(R)$, we have that $f_{i^*}(\tilde{R}_S, R_{-S}) \leq f_{i^*}(R)$. Consequently, we have shown that $f_j(\tilde{R}_S, R_{-S}) \leq f_j(R), \forall j \in N \setminus S$ if $\sum_{i \in S \setminus \{i^*\}} p(\tilde{R}_i) + \sum_{i \in S \setminus \{i^*\}} p(R_i) \leq \Omega$. Finally, let $\sum_{i \in S \setminus \{i^*\}} p(\tilde{R}_i) + \sum_{i \in S \setminus \{i^*\}} p(R_i) > \Omega$. Then $i^*$ gets $f_{i^*}(\tilde{R}_S, R_{-S}) = 0 \leq f_{i^*}(R)$. In addition, by efficiency $f_j(\tilde{R}_S, R_{-S}) \leq p(R_j) = f_j(R)$ for all $j \in N \setminus (S \cup \{i^*\})$. This is what we are looking for.

Claim 2: If $\sum_i p(R_i) \geq \Omega$ and $\sum_{i \in S} f_i(R) = \sum_{i \in S} f_i(\tilde{R}_S, R_{-S})$, then $f_j(R) = f_j(\tilde{R}_S, R_{-S})$ for each $j \in N \setminus S$.

Proof of Claim 2: Set $S^* = \{j \in N \setminus S : f_j(R) \neq f_j(\tilde{R}_S, R_{-S})\}$. If $S^* = \emptyset$ then we are
done. Suppose $S^* \neq \emptyset$ which means that the claim is false. Let $R^1_{N \setminus S^*}$ and $R^2_{N \setminus S^*}$ be such that

$$p(R^1_i) = f_i(R) \& p(R^2_i) = f_i(\tilde{R}_S, R_{-S}), \ \forall i \in N \setminus S^*.$$ 

By repeatedly using Lemma 4.1(b) we obtain that

$$f(R^1_{N \setminus S^*}, R_{S^*}) = f(R) \& f(R^2_{N \setminus S^*}, R_{S^*}) = f(\tilde{R}_S, R_{-S}). \quad (6)$$

By efficiency, $f_i(R) \leq p(R_i)$. If $f_j(R) = p(R_j)$ for all $j \in S^*$, then efficiency and feasibility imply $f_j(\tilde{R}_S, R_{-S}) = p(R_j) = f_j(R)$ for all $j \in S^*$ which contradicts that $S^* = \emptyset$. By feasibility, $\sum_{j \in S^*} f_j(R) = \sum_{j \in S^*} f_j(\tilde{R}_S, R_{-S})$. Consequently, it cannot be the case in which $f_j(R) < f_j(\tilde{R}_S, R_{-S})$ for each $j \in S^*$. Thus, for some $j \in S^*$, $f_j(\tilde{R}_S, R_{-S}) < f_j(R) \leq p(R_j)$. By efficiency, we then have that $f_i(\tilde{R}_S, R_{-S}) \leq p(R_i)$ for all $i \in N$. Consequently, we have that

$$\min\{f_j(R), f_j(\tilde{R}_S, R_{-S})\} < \max\{f_j(R), f_j(\tilde{R}_S, R_{-S})\} \leq p(R_j) \ \forall j \in S^*. \quad (7)$$

Let $\tilde{R}_{S^*}$ be such that

$$p(\tilde{R}_j) = \frac{f_j(R) + f_j(\tilde{R}_S, R_{-S})}{2}, \ \forall j \in S^*.$$ 

Due to (7), we have

$$\min\{f_j(R), f_j(\tilde{R}_S, R_{-S})\} < p(\tilde{R}_j) < \max\{f_j(R), f_j(\tilde{R}_S, R_{-S})\} \leq p(R_j) \ \forall j \in S^*. \quad (8)$$

In addition,

$$\sum_{j \in S^*} p(\tilde{R}_j) = \sum_{j \in S^*} f_j(R) = \sum_{j \in S^*} f_j(\tilde{R}_S, R_{-S}).$$

Consider now $(R^1_{N \setminus S^*}, \tilde{R}_{S^*})$ and $(R^1_{N \setminus S^*}, \tilde{R}_{S^*})$. By construction, $\sum_{i \in N \setminus S^*} p(R^1_i) + \sum_{i \in S^*} p(R_i) = \Omega$ and $\sum_{i \in N \setminus S^*} p(R^2_i) + \sum_{i \in S^*} p(R_i) = \Omega$. Thus, by efficiency,

$$f_j(R^1_{N \setminus S^*}, \tilde{R}_{S^*}) = f_j(R^2_{N \setminus S^*}, \tilde{R}_{S^*}) = p(\tilde{R}_j), \ \forall j \in S^*.$$ 

By combining the equation above, (6) and (8), we find that for each $j \in S^*$ with $f_j(R) > f_j(\tilde{R}_S, R_{-S})$,

$$f(R^2_{N \setminus S^*}, \tilde{R}_{S^*}) P_j \ f(R^2_{N \setminus S^*}, R_{S^*}).$$

Similarly, for each $j$ with $f_j(R) < f_j(\tilde{R}_S, R_{-S})$ then

$$f(R^1_{N \setminus S^*}, \tilde{R}_{S^*}) P_j \ f(R^1_{N \setminus S^*}, R_{S^*}).$$

The two relations above mean that $R_{S^*} \in \hat{R}_{S^*}$ contradicting that $f$ is robust group
strategy-proof.

**Claim 3:** If \( \sum_i p(R_i) \geq \Omega \) and \( \sum_{i \in S} f_i(\tilde{R}_S, R_{-S}) > \sum_{i \in S} f(R) \) then \( f_j(\tilde{R}_S, R_{-S}) \leq f_j(R) \) for all \( j \in N \setminus S \).

**Proof of Claim 3:** Suppose otherwise. By efficiency, we know that \( f_i(\tilde{R}_S, R_{-S}) \leq p(\tilde{R}_i) \) for all \( i \in S \). Fix \( R^1 \) be such that

\[
p(R^1_i) = f_i(\tilde{R}_S, R_{-S}) \quad \forall i \in S
\]
\[
R^1_i = R_i \quad \forall i \in N \setminus S.
\]

By repeatedly using Lemma 4.1(b), we obtain that

\[
f(R^1) = f(\tilde{R}_S, R_{-S}). \quad (9)
\]

In addition, \( p(R^1_i) = f_i(R^1) \) for any \( i \in S \). Let \( R^2 \) such that

\[
p(R^2_i) \leq p(R^1_i) \quad \forall i \in S
\]
\[
\sum_{i \in S} p(R^2_i) = \sum_{i \in S} f_i(R) \quad \forall i \in S
\]
\[
R^2_i = R_i \quad \forall i \in N \setminus S.
\]

We now reach \( R^2 \) from \( R^1 \) by sequentially changing the preferences of those in \( S \). Because \( \sum_{i \in S} p(R^2_i) = \sum_{i \in S} f_i(R) \) and \( f_i(R) \leq p(R_i) \) for all \( i \in N \), at any step of this process, there will be an over-demand. Therefore, by efficiency and strategy-proofness, the allocation of the agent whose preference peak decreases must decrease to her new peak. Then by replacement monotonicity, the allocation of those in \( N \setminus S \) must weakly increase. Hence, we find that

\[
f_j(R) = f_j(R^2) \geq f_j(R^1) = f_j(\tilde{R}_S, R_{-S}) \quad \forall j \in N \setminus S \quad (10)
\]

and

\[
p(R^2_i) = f_i(R^2) \quad \forall i \in S.
\]

By construction, \( \sum_{i \in S} p(R^2_i) = \sum_{i \in S} f_i(R) \) and \( R^2_{-S} = R_{-S} \). Thus, by Claim 2 we must have

\[
f_j(R^2) = f_j(R), \quad \forall j \in N \setminus S.
\]

By combining the equation above, (9) and (10), we complete the proof. \( \square \)